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Fuzzy-based linguistic patterns as a tool for the flexible assessment of a priority vector obtained by pairwise comparisons

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Abstract

A novel approach to the assessment of a hierarchy vector generated by an expert pairwise comparison matrix is proposed. The 'quality' of a given hierarchy is measured using so-called 'linguistic patterns', i.e., specially constructed logical expressions, and an expert assessment matrix. The idea of this approach is based on a concept proposed previously as a flexible criterion for the domain of facility layout problems. The adaptation of the linguistic pattern approach to the assessment of a hierarchy vector also (by analogy to the analytic hierarchy process or AHP) facilitates evaluations of expert consistency. The proposed idea is shown to be less sensitive to potential expert errors than both the classical and fuzzy AHP techniques. The proposed approach is described in detail and illustrated with simple examples. Some features of the proposed approach were studied using computer simulation experiments and the results are reported together with the conclusions obtained.

Keywords: Analytic hierarchy process; Facility layout problem; Fuzzy set theory; Linear ordering; Linguistic variable; Multivalued logic; Possibility theory

1. Introduction

Pairwise comparisons are used widely to elicit personal preferences. The research results obtained using pairwise comparisons are typically represented as a square matrix. This notation is also used to obtain a hierarchy of preferences. In general, it is also possible to determine whether the pairwise evaluations are consistent, which means that the following transitivity relationship is preserved: if object A is preferred over B, and B over C, then A should also be preferred over C.

In the 19th century [31], researchers started using pairwise comparisons for subjective assessments when two physical stimuli differed in terms of intensity. Thus, various types of stimuli can be compared in pairs and with different ranges of intensities, thereby obtaining the following simple relationship: $dI/I = K = \text{const}$. The so-called "just-noticeable difference" depends on the magnitude of the stimulus compared I, which was formalized as the

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Webber–Fechner law to describe the relationship between a psychological sensation (S) evoked by the physical intensity of a stimulus (I): $S = K \log(I)$, where K is a constant that depends on the character of the stimulus [31]. The scientific interest in determining the hierarchy of objects based on pairwise comparisons continued in the 1920s with the development of empirical psychology. The most important research in this field was initiated by Thurstone [39]. His model known as the "Law of Comparative Judgment" was based on investigations of the possibility of measuring purely subjective properties such as the strength of an individual's opinions or feelings about various matters and the subjectively perceived quality of things. This trend in empirical psychology was an extension of the aforementioned studies of the perception of physical stimuli, which can be measured objectively. A further development of the "Law of Comparative Judgment" concept led to the construction of the currently employed version, which is the logistic model referred to as Thurstone's Comparative Judgment (CJ) model. The essence of this extension is based on an assumption that the difference between the values of hidden objects is determined by the dominance of one object compared with another, which affects a person in a logarithmic manner, similar to the Webber–Fechner law. A specific area where the CJ model is applied is educational assessment. A modified version of the CJ approach was described by Pollit [29] and Scher [35].

The Webber–Fechner law motivated the construction of the scales used in the Analytic Hierarchy Process (AHP) method developed by Saaty [32–34], where a numerical ratio scale was assigned to nine linguistic expressions and used to find a hierarchy vector with the eigenvector technique. This method also facilitates assessments of the consistency among experts. AHP belongs to a group of highly popular and comprehensive approaches that utilize pairwise comparisons. AHP is applied in fields where expert knowledge is used to make decisions based on multiple criteria. For example, an extensive review of applications to marketing was provided previously [8,11]. Another study [37] presented a review of industrial applications of AHP based on 290 articles published between 1990 and 2009. This analysis showed that AHP has been applied mainly to various macro-problems (complex and real), as well as managerial and subjective-oriented problems.

Pairwise comparisons have also been used to develop methods that are not based directly on psychophysical research. For example, some employ the nonmetric multidimensional scaling technique proposed by Kruskal [21]. Psychophysical justifications of expert behavior are also ignored in the general linear ordering problem (LOP), which focuses mainly on procedures or algorithms for determining the optimal linear order vector based on pairwise comparisons, although there are no specific requirements about the scales applied and the methods used making judgments. The evaluation of the quality of the vector obtained is general and it does not refer to the ranks assigned to individual objects. This brief survey of the historical and applied aspects of searching hierarchies based on pairwise comparisons demonstrates that a great variety of techniques are employed and there are many applications areas for this approach.

It appears that the popularity of this approach is related to the simplicity of comparing only two alternatives at a time, as well as its empirically supported effectiveness. For example, Koczkodaj [19] demonstrated the substantial advantages of pairwise comparisons for the valid direct ranking of investigated objects.

This brief literature review shows that two main types of method dominate the modeling of hierarchy searching problems based on pairwise comparisons. The first type is the classical LOP, where the quality criterion has a holistic nature. The hierarchy is assessed as a whole and the only concern is the relative positioning of the objects. The second type comprises methods that rank objects based on the means of numbers defined on scales. The most popular of these methods is the AHP and its modifications. The main objective when establishing a hierarchy is exactly the same according to both approaches, but they employ different measurement scales for their comparisons and optimality criteria; thus, the methods used to search for optimal hierarchies are also different.

1.1. Classic LOP

In a classical LOP, the measurement scale is not defined in detail and it is assumed that it can be any ordinal scale. The optimal hierarchy vector in this model is a permutation (p) of objects such that the preference matrix (A) sorted according to p has a maximal sum of assessment values located above the diagonal.

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathrm{A}_{p^*(i)p^*(j)} \rightarrow \max \tag{1}$$

The interpretation of this criterion is particularly obvious for zero-one comparisons, i.e., if object i is preferred over object j, the value one appears in the ij cell of the comparisons matrix, otherwise the ij cell contains zero. If this matrix is sorted according to the optimal hierarchy, every "one" above the diagonal denotes a correct location of the object pair in the hierarchy vector. In addition to the zero-one values in the comparisons, the matrix can be treated as a generalization of this interpretation, although this may lead to other solutions compared with those obtained when only the order of objects in a vector is considered (De Vries [40]). Occasionally, the classical criterion formula (1) is modified by multiplying the preference values by a monotonic function $f(j-i)$ in order to consider the distance between a given pair in the hierarchy. This change yields somewhat different solutions compared with the classical criterion.

As mentioned earlier, the pairwise assessment matrix can be analyzed to assess consistency, which is understood as the degree of transitivity of preferences. The most common way of assessing this aspect of the relationship is to study three-element cycles in a directed graph that represents the matrix. The coefficient *Tau* proposed by Kendall [17] measures the matrix inconsistency based on the ratio of the number of three-element cycles occurring in the array relative to the number of all theoretically possible cycles for a matrix of a given size. Depending on the context, an inconsistency may occur due to the nature of the phenomenon studied or due to the inconsistency of the expert making the evaluation. In general, high inconsistency makes it difficult to find the vector of preferences.

The search for an optimal, in the sense of criterion (1), vector of preferences in the classical LOP is based mainly on various types of heuristics. An extensive review of the diverse classes of approaches and algorithms employed was provided by Marti and Reinelt [23]. These authors discussed analytical approaches such as *branch and bound* and *cutting plane* techniques, but their limited efficiency means that these methods are of less importance than the approximate algorithms that dominate this area. In addition to heuristic methods based on local neighborhood search or construction methods, the most common approaches are metaheuristics, including genetic algorithms and simulated annealing.

The consistency (transitivity) assessment for a pairwise comparisons matrix does not form part of the LOP model and it is not incorporated into the hierarchy search algorithms. Kendall's idea and its modifications occur only in specific applications of pairwise comparisons, e.g., in the field of animal behavior (De Vries [40]). However, these comparisons are derived from objective data, e.g., based on the results of fights observed between individuals. Thus, the consistency examination is used to determine the structure of dominance in groups rather than analyzing the coherence of the matrix itself.

1.2. Classic AHP

Thirty years ago, a competitor to the classical LOP approach was proposed by Saaty [32], AHP, which is also based on pairwise comparisons, and it has numerous applications. The AHP attempts to find the hierarchy vector based on the comparison matrix eigenvector associated with the largest eigenvalue. This idea is similar to the conceptual basis of principal components analysis, but instead of reproducing correlation or covariance matrices, the hierarchy vector reconstructs the comparison matrix. Thus, the optimality criterion is the extent to which the matrix is reproduced by a given vector. Saaty proved that the eigenvector that corresponds to the largest eigenvalue reconstructs the best matrix, so the analytic nature of this approach involves the determination of this vector. However, to use this approach, one needs to assess the variants on a ratio scale and the comparisons matrix must be reciprocal ($a_{ji} = 1/a_{ij}$).

The analysis of consistency (transitivity) plays an important role in AHP [32,33]. Compared with the evaluation of transitivity using Kendall's Tau coefficient, the consistency ratio used in AHP is more restrictive. It includes the direction of the preference relationships but also their relative intensities. In practical applications, it is even recommended that the expert's consistency is monitored during the assessment process to detect and correct the most incoherent scores.

One of the claimed weaknesses of the AHP approach is the special method used for expressing expert opinions about the preferences for alternatives compared in pairs. It is assumed that the assessment of the degree of preference for one alternative over another is the ratio (quotient) of their *usabilities*. This method of evaluating the results obtained by the mathematical procedure depends only on the reciprocal matrix and the hierarchy can be determined as an eigenvector. Saaty [32] argues that this scale is justified by the Webber–Fechner law, as generalized later by Stevens [36]. This formula describes the relationship between a subject's subjective impression and the intensity of a stimulus. However, this approach has been empirically supported only in the context of physical effects. As documented in studies in the field of psychophysiology (e.g., Proctor and Van Zandt [31]), the application of this ratio scale to the description of relationships between subjective feelings and a stimulus level is limited only to some measurable physical quantities such as mass or sound. Indeed, the assumption that an expert assesses the preferences as a quotient of the usabilities of alternatives is debatable in the case of more abstract concepts.

Another moot issue regarding the AHP method is the inconsistency between the *soft* nature of human judgments and the highly accurate method used to assess the hierarchy of elements obtained from the eigenvector. Frequently, digits occur in the third or fourth place after the decimal point, which determine the exact positioning of alternatives in the hierarchy, but a subjective assessment is usually much less precise. This discrepancy makes the AHP method relatively sensitive to errors during assessments. Detailed empirical research on this subject was presented by Gass and Standard [13].

The concerns presented above, and a few others, led to the appearance of a more flexible formulation for pairwise comparisons ratings, and thus different approaches to the modeling of the assessments obtained. These approaches can be regarded as efforts to make the AHP search for a hierarchy based on pairwise comparisons in a more flexible manner [5].

1.3. Fuzzy set-based methods

Methods based on approximate assessments are focused mainly on the use of fuzzy sets theory. One of the main trends involves fuzzy AHP-based methods. Among the various methods proposed, the most important approaches employ direct fuzzification of Saaty's concepts. In this approach, the eigenvector method is used to determine the hierarchy for interval data obtained using alpha-cuts. This direct fuzzification of Saaty's technique was described by Buckley [2,3], who proposed the fuzzy numbers concept as a basis for expressing preferences. In fact, the hierarchy is calculated by applying the traditional determination of eigenvectors to suitable alpha-cuts of fuzzy numbers. The final hierarchy is obtained by comparing the fuzzy numbers in the resulting vector. The problem of the consistency of ratings in this method is also based on traditional AHP computations.

A similar approach was presented by Cheng and Mon [10], who introduced an optimism indicator that allows hierarchies to be calculated for single values of intervals for individual alpha-cuts. These values are set inside the interval as a weighted mean by a proposed indicator with values ranging from 0 to 1.

Competitive approaches to the fuzzification of AHP are based on other eigenvector techniques for obtaining hierarchy vectors. However, in these approaches, the methods used generally calculate the vectors for the intervals obtained based on the alpha-cuts of fuzzy sets that represent the pairs compared in assessments. One of the first methods of this type was proposed by Laarkoven and Pedrycz [21], who determined the vector of preferences for triangular fuzzy sets by logarithmic regression. A similar approach was developed by Mikhajlow [27], who suggested a solution based on a linear programming model, where alpha-cuts were also used to decompose the comparisons of fuzzy number ratings into numerical intervals. Next, using fuzzy linear programming, the strict numerical values of the preferences were found for every alpha-cut. Aggregating the values obtained for the alpha-cuts yielded the final priority vector. The optimality criterion used in both of these approaches aims to minimize the distance between the assessment matrix and the matrix that is reconstructed. The aggregation of the values obtained for alpha-cuts provides the final priority vector. However, these approaches do not address the consistency issue.

The search for hierarchies based on fuzzy numbers arithmetic forms a separate research area. Buckley and Uppuluri proposed the first method for determining the hierarchy as a geometrical average of the comparisons matrix, which contains scores in the form of fuzzy numbers [3]. This concept has gained significant popularity. The relative simplicity of its implementation compared with the eigenvector approach or calculating preferences using alpha-cuts means that this approach is treated as the de facto standard in studies concerned with fuzzy AHP applications [20,37].

The concept proposed by Deng [11] also belongs in this area, where he developed a very simple approach based on the expert opinions represented by fuzzy numbers. This method is a variation of the so-called naïve use of AHP, which involves the aggregation of all the preferences in the rows of the comparisons matrix. The summation is obtained according to the rules related to fuzzy numbers, although the issue of the consistency among ratings was not addressed. The often overlooked problem of coherency in a comparisons matrix seems to be important in the fuzzy AHP group of methods. Buckley and Uppuluri [3] showed that AHP based on fuzzy numbers arithmetic provides outcomes that are comparable with the classical AHP method for consistent matrices. Coherent results of comparisons also eliminate errors (Buckley at al. [4]), as noted in the method proposed by Laarkoven and Pedrycz [22].

An interesting approach in this area was also suggested by Maa et al. [23], who proposed a procedure for repairing the pairwise comparisons matrix filled by an expert. The matrix contains membership function values for the preference relationships. The application of the proposed method means that the matrix has weak transitivity, but the repaired matrix allows the determination of the hierarchy using simple aggregation methods, e.g., by summing the items in consecutive matrix rows. In this procedure, the solution criterion and transitivity examination are considered simultaneously because the resulting hierarchy corresponds to the repaired matrix, thereby satisfying the transitivity property matrix.. Similar approach to repair consistency matrix has been proposed lately in the work of Xia and Hen [42]. However, *fixing* the expert's ratings seems to be an artificial intervention, especially if the intransitivity is derived from the nature of the phenomenon. These fuzzy set-based approaches aim to obtain greater flexibility with respect to the expert ratings and they reduce the discrepancy between the imprecision of subjective assessments and the numerical precision of the AHP method. The problem of the transitivity relationships of fuzzy sets (numbers) has been analyzed often previously, but it is frequently abandoned in studies that deal with fuzzy-based versions of AHP. Various types of transitivity were systematized by Wang [41], as well as by Gogus and Boucher [14].

1.4. Current study objectives

The main goal of this research was to construct a method for determining a hierarchy based on pairwise comparisons by exploiting the flexibility of fuzzy approaches and by mitigating some of the problems found in both classical AHP and its fuzzy modifications (compare [4]). In particular, the proposed concept combines fuzzy assessments and the LOP perspective, and it has the following properties:

- It does not require complex computations to obtain the hierarchy vector, unlike earlier approaches;
- It does not make any assumptions regarding the mechanisms of human reaction to immeasurable stimuli, as discussed in the introductory section;
- In contrast to the methods based on fuzzy numbers arithmetic, the consistency of the expert assessments matrix is not necessary, although it permits their estimation;
- It is not sensitive to the absence of assessments (unlike all AHP-based approaches);
- It is less vulnerable to changes (mistakes) in the assessment matrix than existing methods.

The concept of linguistic patterns (LPs) introduced by Grobelny [16] was applied to the following two issues: assessing the optimality of the hierarchy vector and evaluating the pairwise comparison ratings of experts. In the present study, the proposed method allows an assessment of the hierarchy vector relative to the matrix of preferences as the multi-valued degree of truth of the logical expressions, similar to the fuzzy sets-based approaches described in the previous section, where the vague linguistic expressions of experts are used as preference ratings. The proposed method aims to make the evaluation process similar to the natural functioning of an expert. Analogous to the consistency ratio obtained from the AHP and/or Kendall's Tau from LOP, the proposed approach also allows the approximation of the degree of consistency (transitivity) among the imprecise preferences ratings. The application of these ideas to an expert pairwise evaluation reduces the sensitivity to assessment errors for the hierarchies obtained (Gass and Standard [13]). This study describes the details of the proposed method, as well as the outputs obtained based on simulation studies.

The next section of this paper explains the idea of LPs in its first application to the facilities layout problem (FLP) domain. Next, we demonstrate the possible application of the proposed concepts to a specific FLP type of problem, i.e., "multi-product line with unidirectional flow". The layout optimization in this case requires the determination of a hierarchy and it is reduced to LOP. The LOP expressed in fuzzy-based LPs is considered deeply in Sections 4 and 5. The proposed method for assessing the consistency of expert opinions is also provided, as well as the results of simulation experiments that tested the sensitivity of the expert assessment matrix to changes. The results are compared with those obtained using the traditional AHP method and its fuzzy equivalent.

2. Linguistic pattern as a criterion in an FLP

As mentioned above, the LP has been proposed as a criterion when modeling the FLP (Grobelny [15]). The LP is defined as a logical sentence that combines linguistic variables. Linguistic variables encapsulate the values of verbal expressions in a similar manner to a natural language. A logical expression is a construct that can be assessed as true

	a	b	c
a	-	50	40
b		-	30

a	c	b	Layout 1
c	b	a	Layout 2

Fig. 1. Example based on links as the number of transport operations between each pair of facilities serviced in the sequence *ij* or *ji*, and two layouts of facilities a, b, c on a modular grid.

or false. A typical example of a logical expression is the implication that if A then B, where A and B are expressions that are similar to a natural language, which can also be evaluated as true or false.

A very simple definition of an FLP can be summarized as follows. Given N objects, we want to put them on a given plane such that they minimize the following relationship:

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathrm{L}_{ij}\mathrm{D}_{ij} \rightarrow \min, \tag{2}$$

where L_{ij} is a link level (strength of interrelation) of the i–j pair and D_{ij} denotes the distance between the elements of this pair.

In industrial engineering, this relationship (2) is usually interpreted as the minimization of the overall transportation distances (costs) in a system of N connected machines. For example, if devices need to be arranged on an assembly line, we may represent the available locations in the form of a modular grid with unitary distances, as shown in Fig. 1. Assuming that the matrix presented in Fig. 1 contains the number of transport operations for each pair of facilities, we can compute the overall transportation costs for the two possible layouts. Based on relationship (2) we obtain $50 \times 2 + 40 \times 1 + 30 \times 1 = 170$ for the first layout, and $50 \times 1 + 40 \times 2 + 30 \times 1 = 160$ for the second. Thus the second option is better in terms of criterion (2). Naturally, this optimization involves searching for the solution with the lowest value of (2) in the given conditions.

Obviously, the L_{ij} matrix in FLP tasks is derived from pairwise comparisons of the analyzed objects, where its content depends greatly on the context. For example, in industrial engineering problems, the relationships derive from the technology applied, which are given as actual data, such as the number of transport operations between pairs of objects and/or the unit costs of these operations. However, in many other fields, this approach is either impossible or very limited. Thus, since the early 1960s, subjective expert-based methods for determining relationships have increased significantly. One of the main methods proposed in this area was elaborated by Muther [28] in his Systematic Plant Layout (SLP) concept, where L matrix values are obtained using Relation Charts. An expert present in these charts gives their subjective opinions about the necessity of the adjacency of objects and the importance of this adjacency in a form of linguistic expressions: A – absolutely necessary, E – important, O – ordinary closeness OK, U – unnecessary, X – not desirable. In the SLP approach, these phrases are replaced with natural numbers in further mathematical analyses, and Muther's idea of allowing experts to make assessments in natural language expressions inspired the LP concept described in the present study.

The general form of the LP for formula (2) can be determined based on some expressions in everyday language. It is reasonable to suggest that a good layout for a given set of objects satisfies the following relationship: "All pairs of strongly related objects are adjacent in the layout". After slightly formalizing this relationship, we can obtain formula (3).

$$\text{IF link\_between\_objects}(i, j) \text{ is STRONG THEN distance\_between \_objects}(i, j) \text{ is SMALL} \tag{3}$$

The statement above is an example of an LP. This is an implication, so we can assess the truth of this statement, i.e., knowing the truth of the left- and right-hand sides of the sentence using a classical implication truth table. It is obvious that (3) is false only for the situation where two objects have a STRONG link and they are far from each other in a given arrangement. Both expressions ("link_between_objects is STRONG" and "the distance_between_objects is small") are not very precise, so the truth of the above should be assessed in a rather flexible manner. Grobelny [16] proposed the use of a multivalued logic Lukasiewicz formula to evaluate the truth value of (3), and some concepts from the possibility theory proposed by Zadeh [43] to assess the level of truth for the left- and right-hand sides of (3). These assumptions led to the following formula for evaluating one specific object pair in a given layout.

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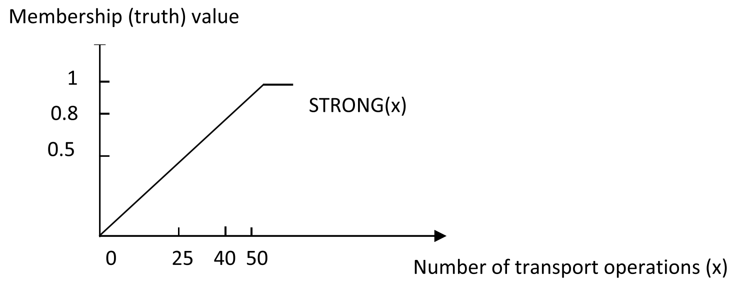

Fig. 2. STRONG pattern expressed as a fuzzy set in the universe of discourse proposed for the example illustrated in Fig. 1.

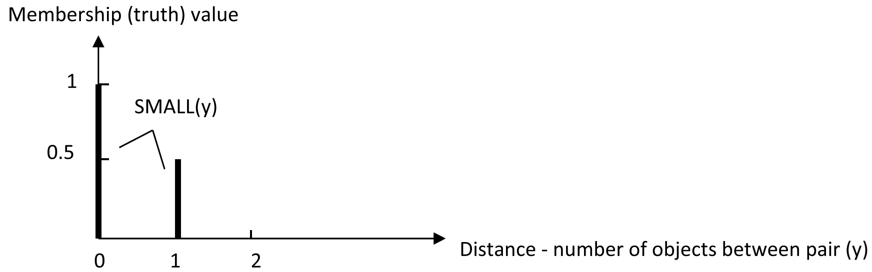

Fig. 3. Fuzzy set that represents the SMALL pattern expression in the universe of discourse for the distance, which is defined as the number of objects between the pair analyzed.

$$\text{Truth\_of (3)} = \min\big(1, 1 - \text{truth\_of (l)} + \text{truth\_of (r)}\big) \quad (4)$$

The Lukasiewicz formula allows the determination of the implication truth value, which is expressed as a number in the range [0, 1], and it is based on the "degrees of truth" for the left- and right-hand sides. It is easy to see that this designation of the truth value is a generalization of the classical implication truth table.

Thus, (4) determines the truth value level of a criterion (3) for a given pair of objects (i, j). In the present study, the terms "the truth_of (l)" and "the truth_of (r)" denote "the degree of satisfying criteria for a given pair of facilities in a given layout" for the left- and right-hand side of the pattern (3), respectively, i.e., for the expressions "link_of_objects is STRONG" and "distance_of_objects is SMALL." In the remainder of this study, we use the following designations interchangeably: degree of truth, truth value, level of truth value, etc., to refer to different ways of determining the degrees of compliance/matching between fuzzy data and their criteria, as suggested by Zadeh [44], in terms of the relative truth of proposition p with respect to reference proposition r.

These "truth levels" can be determined in many different ways. The simplest and most intuitive method is to gather expert opinions that directly express the level of truth for each pair of objects. In the context considered in Fig. 1, it is relatively easy to obtain these data for both sides of pattern (3). An industrial engineer may formulate his understanding of STRONG and SMALL for links and distances between objects, respectively, in the form of fuzzy sets by specifying the levels of truth regarding the fulfillment of appropriate criteria by measured physical values.

As shown in Fig. 1, the links are physically measurable so a linear function of the level of truth appears to be a reasonable approach for determining the truth_of (l). This function can be defined within the range of [0, 1], i.e., proportionally from 0, if there are no transport operations, to 1, which denotes the maximum possible number of transport operations. When dealing with the discrete values of distances between the locations of facilities, a similar relationship may be proposed in the form of singletons. These concepts are illustrated in Figs. 2 and 3, respectively.

It is more comfortable to express opinions using linguistic categories similar to a natural language, so we may also postulate other ways for determining the truth levels of expressions provided in this manner. If an expert expresses an opinion about the relationship between the objects in BIG, SMALL, MEDIUM, etc. categories, then it is intuitive to assign a truth value of 1 to a statement that describes the biggest category and a small truth to the lowest score, with proportional values for the remaining categories. Thus, representing linguistic terms using regular fuzzy set variables (e.g., triangular numbers) and a linear membership function (Fig. 4) also appears to be intuitive.

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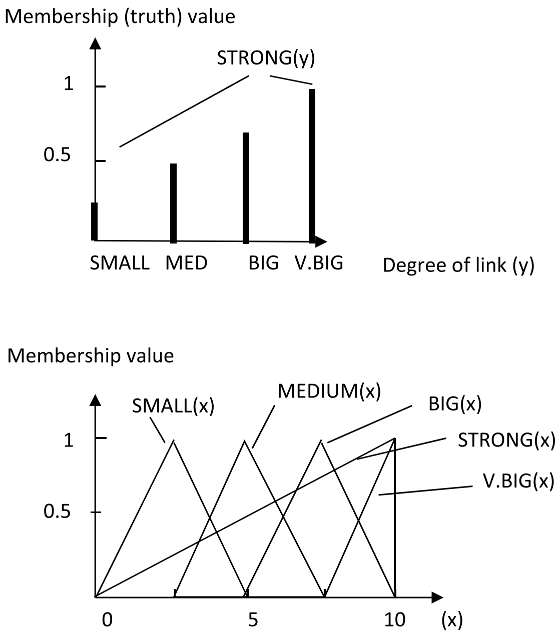

Fig. 4. Intuitively defined fuzzy sets that represent expert opinions by directly assigning the truth values (upper graph), and regular fuzzy sets in the form of triangular numbers in the artificial universe of discourse (lower graph).

This approach is based on the fact that even linear distances between intensities are probably the most natural. However, we should remember that specifications of the intensity degree using linguistic expression are ordinal in nature and the real distances between them are not known.

More formal approaches for determining the degree of truth regarding the expressions represented by fuzzy sets in various applications areas are actually generalizations and extensions of the intuition described above. Zadeh [44] introduced two different ways of specifying the degree of truth for the situations discussed in the present study: CONSISTENCY and COMPATIBILITY. Both approaches are based on a "relative truth" paradigm regarding expressions such as: L is A with respect to a reference proposition that L is R. The first measure, i.e., CONSISTENCY, appears to be appropriate for discussing context because it gives the "truth value" as a number from the range of 0–1. COMPATIBILITY allows us to obtain the truth degree as a fuzzy set, which can be approximated based on a linguistic assessment of the types: "true," "very true," "not true," etc., but this cannot be applied directly in the context considered in the present study. CONSISTENCY is computed as an appropriate possibility value (Zadeh [43,44]: CONS(L is R, L is A) = POSS (L is R I L is A) = $\mathrm{Sup}_x(\min(\mathrm{R}(x), \mathrm{A}(x)))$, where X is the universe of discourse, and $\mathrm{R}(x)$ and $\mathrm{A}(x)$ are the membership functions of R and A, respectively. Thus, for pattern (3), we may write the following.

$$\text{Truth\_of (l)} = \text{POSS}\big(\text{link\_of\_objects(a, b) is } \textbf{STRONG} | \text{link\_of\_objects(a, b) is } \textbf{A}\big)$$
$$\text{Truth\_of (l)} = \max_x(\min\big(\text{STRONG}(x), \text{A}(x)\big) \tag{5}$$

The truth indicator computed according to formula (5) can provide the (relative) truth degree for both sides of pattern (3). This measure reflects the degree of overlap for the STRONG and A sets.

The term called NECESSITY proposed in the fuzzy pattern matching area [7] may also be used to assess the extent of compliance by A set to the reference proposition R using (5). In this case, relationship (5) takes the following form.

$$\begin{aligned}&\text{NEC}\big(\text{link\_of\_objects(a, b) is STRONG}|\text{link\_of\_objects(a, b) is } \textbf{A}\big)\\&\quad = 1 - \text{POSS}\big(\text{link\_of\_objects(a, b) is NOT STRONG}|\text{link\_of\_objects(a, b) is } \textbf{A}\big) =\\&\quad = \mathit{inf}_x\big(\max\big(\text{STRONG}(x), 1 - \text{A}(x)\big)\big)\end{aligned} \tag{6}$$

The measure (6) is interpreted as *certainty* and it specifies the extent to which the sets NOT STRONG and A do not overlap. Measures (5) and (6) are associated with optimistic and pessimistic assessments of the compliance of A values

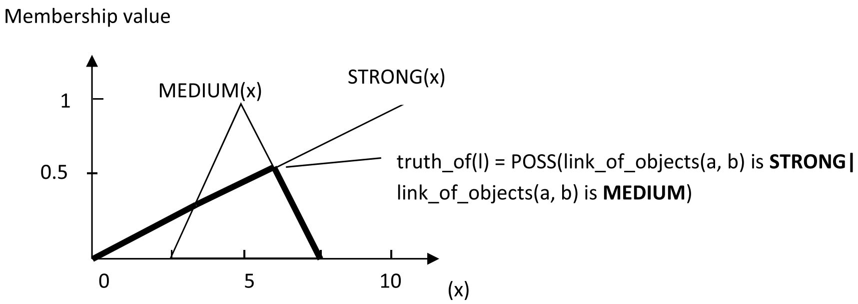

Fig. 5. Graphical illustration of possible definitions of linguistic variables and the calculation of t(l) according to equation (5).

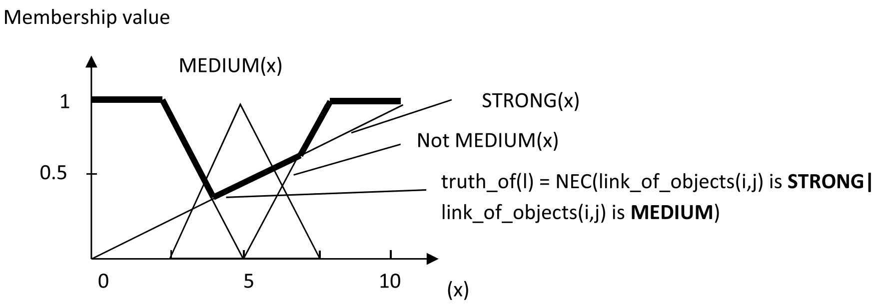

Fig. 6. Graphical illustration of possible definitions of linguistic variables and the calculation of t(l) according to equation (6).

with the requirements of R, respectively. Moreover, in the framework of fuzzy pattern matching, Bouchon-Meunier et al. [1] presented another version of the aforementioned COMPATIBILITY indicator, which is simply an arithmetic mean of measures (5) and (6).

$$\text{COMP(L is STRONG, L is A)} = \big(\text{POSS(L is STRONG I L is A)} + \text{NEC(L is STRONG I L is A)}\big)/2 \qquad (7)$$

When considering fuzzy pattern matching, we do not use the truth_value term for the measures described. However, it seems that this idea is consistent with the aims of Zadeh's [44] proposals regarding his definition of CONSISTENCY, which he uses to compute the relative truth level for the expression "L is A" with respect to "L is R".

Examples of the methods for determining POSSIBILITY and NECESSITY measures are illustrated in Figs. 5 and 6, respectively.

The exact values of POSSIBILITY and NECESSITY from Figs. 5 and 6 are 0.6 and 0.4, respectively, thus the compatibility measure in this example, which is computed according to (7), equals 0.5. Similarly, the compatibility indicators for the SMALL and V.BIG link values are 0.3 and 0.9, which represent a compromise between the optimistic possibility (0.4 and 1, respectively) and pessimistic necessity (0.2 and 0.8, respectively).

It should be noted that if $A(x)$ is a singleton then both the possibility and the necessity of (L is R, L is $A(x_i) = R(x_i)$), i.e., the assessed degree of the truth value is a membership function value of reference proposition R, as found in the example in Fig. 3.

The possibility of using various definitions of fuzzy sets to represent expert linguistic expressions and reference propositions, as well as diverse compliance measures, can be used in the proposed method to reflect more complex, nonlinear relationships between relative truth and expert opinions. These sophisticated relationships can model a decision-maker's way of thinking about the problem rather than expert opinion, which is usually expressed on an ordinal scale.

In the following examples and simulation studies, the simple linear method was used to define the degrees of truth for the left- and right-hand sides of the pattern (3). This appears to be the most intuitive model of decision-maker thinking in the context considered.

It is quite clear that the evaluation of a given layout should include the pattern (3) truth assessment for all pairs of objects. Grobelny [16] proposed its computation as the "mean truth" for all of the objects pairs in the layout. This measure is simply the sum of the pattern (3) truths for all pairs divided by the total number of pairs, which allows the evaluation of the layout to be expressed on a scale from 0 to 1, where 1 indicates that the system satisfies the formula fully or ideally. Flexible and often imprecisely defined patterns and fuzzy sets can be inconsistent, so a degree of truth equal to 1 could be unreachable in some cases. Grobelny [15] showed how to calculate the upper bound level of the truth value for given (imprecise) data. This procedure allows the theoretical maximum of the "mean truth" to be calculated with (3).

Like many other approaches to using fuzzy sets, the definitions of linguistic expressions are treated in a rather flexible manner. Thus, they can be precisely determined only in specific contexts. In general, as noted above, patterns are extensions of classical reasoning based on an implication. For example in the case of the FLP (Fig. 1), we can determine that objects such as containers with different parts, which need to be rationally arranged on the working plane, are strongly linked if they are serviced one after another more than 30 times per day. We can also specify that a "small distance" means the direct adjacency of the two containers. Given these assumptions, the example depicted in Fig. 1 can be analyzed as follows.

For Layout 1 (a c b), we have:

(a, b) t(l) = 1; t(r) = 0 so Truth_of (3) = 0

(a, c) t(l) = 1; t(r) = 1 so Truth_of (3) = 1.

Similarly, for Layout 2 (c b a), we calculate:

(a, b) t(l) = 1; t(r) = 1 so Truth_of (3) = 1

(a, c) t(l) = 1; t(r) = 0 so Truth_of (3) = 0.

Thus, (using the average value of the truth) layouts 1 and 2 satisfy the LP at the level of 0.5. In the calculations of the "mean truth value" in this example, the pair of (b, c) objects were not considered because the LP only regards pairs where the link between objects is STRONG.

As mentioned earlier, the same result can be obtained using the classical truth value table because the Lukasiewicz implication truth formula is a generalization of this table. It should be noted that this approach for estimating the degree of truth in multi-valued logic is not the only one available. Others, such as Godel equations, have been discussed in Prade [30], but the Lukasiewicz approach was the first in this area historically. In addition, as noted by Zimmerman [45], this formula satisfies the system of axioms for "truth functionality", i.e., the truth of *if p then q* is dependent only on the truths of p and q, respectively.

To benefit fully from the proposed approach, we should define the elements of pattern (3), i.e., the details of the left- and right-hand sides.

Using the definitions from Figs. 2 and 3, we can now calculate the truth of the pattern (3) fulfillment based on the layouts from Fig. 1.

Layout 1

(a, b) Truth_of (3) = min(1, 1 − 1 + 0.5) = 0.5
(a, c) Truth_of (3) = min(1, 1 − 0.8 + 1) = 1
(b, c) Truth_of (3) = min(1, 1 − 0.6 + 1) = 1

Layout 2

(a, b) Truth_of (3) = min(1, 1 − 1 + 1) = 1
(a, c) Truth_of (3) = min(1, 1 − 0.8 + 0.5) = 0.7
(b, c) Truth_of (3) = min(1, 1 − 0.6 + 1) = 1

Now, the average value of the truth for layout 2 (0.9) is slightly higher than that for layout 1 (0.83). The full pattern analysis approach treats all values in a more flexible manner and, in this example, the distance between a given pair can be assessed as "partly small", while the links between objects can also be "partly strong". Clearly, the "pattern" can primarily use approximate expert opinions, which can also be expressed as natural language words.

3. LPs in a hierarchy assessment according to the pairwise comparisons matrix

In general, the search for a hierarchy of objects (linear ordering) based on the preferences obtained from pairwise comparisons is an NP-hard problem. There are two trends in the detailed formulation of the requirements and criteria

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when solving this task. A special feature of these approaches is the method used to express the hierarchy quality criteria. This method, as mentioned earlier, depends mainly on assumptions concerning the scale of comparisons. A review of the scales used in this area was provided by Saaty [32]. In practice, the main difference among the highlighted trends is that when assuming only an ordinal (ordinal) scale, a comparisons rating hierarchy must be evaluated based on a single number assigned to the vector (some criterion value). Thus, the assessment ratio scale adopted in AHP allows numerical priorities to be obtained for each alternative (using an eigenvector in AHP).

The acceptance of a specific assessment scale determines the criteria used to assess the hierarchy (alternative orderings) obtained according to the matrix of expert assessments. In the field of classical problems of linear ordering, this criterion is formulated as follows (Chanas and Kobylanski [9]). Assuming that the matrix $\mathrm{A} - n \times n$, where each element (i, j) represents the value of the preference of i over j, and $p(k)$, a hierarchy vector that best reflects the preferences of an expert used to define this matrix is a permutation $p^*(k)$ of objects $k = 1 \dots N$ such that

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathrm{A}_{p^*(i)p^*(j)} \to \max . \tag{8}$$

This means that the best representation of the expert opinions expressed in the matrix corresponds to the hierarchy of alternatives (permutation of objects) for which the matrix A ordered according to this hierarchy has the maximum sum of preferences above the main diagonal. Some models also suggest multiplying the preferences by the monotonic function $f(j–i)$ to consider the distance between the location of each pair in the hierarchy (Saaty [32]).

Among the many approaches and methods that assume a ratio scale for expert comparisons, the AHP has become very popular over the last two decades. This popularity is due to the method used to find the ordering using eigenvectors and the possibility of assessing the expert consistency (CR), which is a numerical measure of a judgment's transitivity. Although the ideas employed in the present study were inspired by LOP and AHP, the problem of searching for the optimal hierarchy is not of interest. The essence of the proposed approach is to replace criterion (8) by an LP as a criterion for assessing object prioritizations based on preferences that are expressed linguistically. As mentioned above, the proposed approach is based on the analogy of the problem defined by (8) to FLP. Indeed, we may treat relationship (8) as the model of a specific FLP case. Carrie [6] described the systematics of single production line optimization problems. One of the areas identified is the issue of a multi-product line with unidirectional flow, where each product/part is assigned a specific sequence of operations on different individual machines. The layout of the machines in this case should ensure that the objects handled flow in one direction, which is determined by a transport system. The optimization according to criterion (8) yields the solution that minimizes the number of backtracks, and thus the costs. This arrangement is a particular hierarchy of the order of the machines. By using LPs to model this problem, as in relationship (3), we can employ the following simple expression (implication) for a given pair of machines and a given machine's sequence on the production line.

IF a_need_of_precedence_of_machine_i_before_j (*in the assessment matrix*) is STRONG

THEN i_is placed_before_j (*in the machines′ sequence*) (9)

We assume that for each pair of objects in the matrix shown in Fig. 1, several operations for a given machine are specified in a column, which should be preceded by a machine defined in a row. Thus, using the STRONG(x) definition from Fig. 2 (changing only the description of the x axis to "Number of operations in which i precedes j"), it is possible to compute the truth value (9) for layouts 1 and 2 under the assumption of a left-right transport direction.

Layout 1 (a c b)

(a, b) Truth_of (9) $= \min(1, 1 - 1 + 1) = 1$
(a, c) Truth_of (9) $= \min(1, 1 - 0.8 + 1) = 1$
(b, c) Truth_of (9) $= \min(1, 1 - 0.6 + 0) = 0.4$

Layout 2 (c b a)

(a, b) Truth_of (9) $= \min(1, 1 - 1 + 0) = 0$
(a, c) Truth_of (9) $= \min(1, 1 - 0.8 + 0) = 0.2$
(b, c) Truth_of (9) $= \min(1, 1 - 0.6 + 0) = 0.4$

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	a	b	c	d	
					H1- (a b c d)
a	x	BIG	x	x	*H2- (c d b a)*
b	x	x	BiG	V.BIG	
c	MED	x	x	x	
d	SMALL	x	SMALL	x	

Fig. 7. Example of a comparison matrix and two hierarchies.

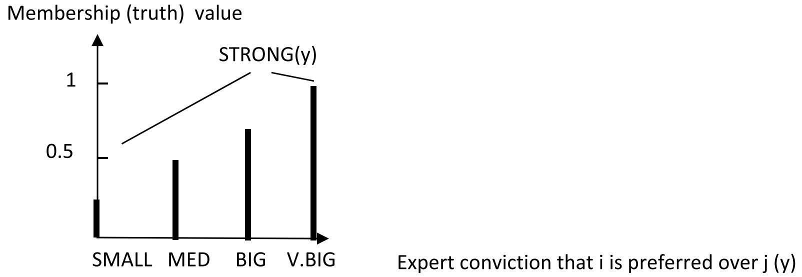


Fig. 8. Example (intuitive) of a method for determining the truth values for the left-hand side of pattern (10).

The mean truth values are equal to 0.8 and 0.9 for layouts 1 and 2, respectively. It is also clear that sequence a–b–c is "fully true" (mean truth value = 1) according to pattern (9).

As in the previous section, it should be noted that the LP approach is quite flexible. For example, the shape of the left- and right-hand expressions must consider the way in which the expert is instructed about the assessment process, particularly the purpose and scope of the comparison scale. For example, the decision-maker might want to obtain a hierarchy based on the level of subjective conviction of an expert according to their preferences (rather than the intensity of this relationship). In this case, (9) can be formulated in the following manner (including the assessment of the object distances in the preference vector):

IF Expert conviction that_a $\succ$ b is STRONG

THEN a_is placed_definitely_before_b (in the preference vector), (10)

where (a $\succ$ b) means a is preferred to b.

For the operationalization of (9) and (10), Lukasiewicz formula (4) can be used with an appropriate technique to determine the truth values for the left- and right-hand sides of the pattern, respectively. Clearly, the linguistic variables and their domains should be defined appropriately for these operations. For example, we assume that an expert comparison matrix filled with linguistic expressions describes their subjective opinion that i is preferred over j for all pairs. The expert's beliefs are expressed by four linguistic expressions: small, medium, big, and very big. Fig. 7 shows an example of the comparison matrix and two examples of hierarchy vectors.

We should determine the expert opinions in a simple and intuitive manner, as shown in Fig. 2. Thus, we assume that the distances between the truth level values are proportional for consecutive intensity levels.

The truth of the right-hand side of pattern (10) is defined in Fig. 8 by assigning the membership function value in proportion to the number of objects that separate the given i–j pair.

The proposed methods require a single comparison of each pair, although it may be natural to compare the objects twice. Thus, in order to determine the truth value for the whole hierarchy, only positive linguistic preferences are considered. As a consequence, the pair evaluation will be shown in the matrix only at position i, j or j, i. It is reasonable to assume that the truth value of the preference j–i is 0 if it takes any positive value for i–j.

Now, we can calculate the truth of pattern (10) for each pair assessed for hierarchies H1 and H2.

For H1 = (a b c d), we obtain:

(a b) t(l) = 0.75 t(r) = 0.5 truth of pattern (10) = min(1, 1 − 0.75 + 0.5) = 0.75
(b c) t(l) = 0.75 t(r) = 0.5 truth of pattern (10) = min(1, 1 − 0.75 + 0.5) = 0.75
(b d) t(l) = 1.0 t(r) = 0.8 truth of pattern (10) = min(1, 1 − 1.0 + 0.8) = 0.8

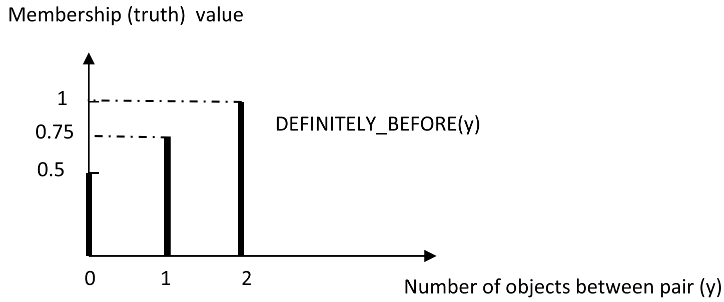


Fig. 9. Definitions of the (intuitive) truth values for the expression "a is placed DEFINITELY before b."

(c a) t(l) = 1.0 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 1.0 + 0.0) = 0.0
(d a) t(l) = 0.25 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 0.25 + 0.0) = 0.75
(d c) t(l) = 0.25 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 0.25 + 0.0) = 0.75.
The mean truth value for the vector H1 is 3.8/6 = 0.63
For H2 = (c d b a), we obtain:
(a b) t(l) = 0.75 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 0.75 + 0.0) = 0.25
(b c) t(l) = 0.75 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 0.75 + 0.0) = 0.25
(b d) t(l) = 1.0 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 1.0 + 0.0) = 0.0
(c a) t(l) = 1.0 t(r) = 1.0 truth of pattern (10) = min(1, 1 − 1.0 + 1.0) = 1.0
(d a) t(l) = 0.25 t(r) = 0.75 truth of pattern (10) = min(1, 1 − 0.25 + 0.75) = 1.0
(d c) t(l) = 0.25 t(r) = 0.0 truth of pattern (10) = min(1, 1 − 0.25 + 0.0) = 0.75.
The calculations above for H2 (c d b a) show that the mean truth value is equal to 0.54, thus H1 is slightly better according to the proposed pattern and the linguistic variable definitions.

4. Analysis of the preference matrix properties

The approach proposed in this study aims to calculate the truth value for a given LP with a specific hierarchy vector and a given matrix of pairwise comparisons. Compared with the classical AHP approach, it is clear that the proposed method is very "soft" and it is based on very weak assumptions.

These assumptions are weaker than previously proposed methods for fuzzy AHP (e.g., Buckley and Uppuluri [3], Buckley et al. [4], Deng [12]) or the fuzzy analysis of preferences (Maa et al. [23]). We only assume that the fuzzy sets representing linguistic expressions should be built depending on the context and in a reasonable manner. However, we can suggest some tools for characterizing the matrix of preferences (or rather an expert who determines the matrix) in a similar manner to the concepts of the CR and/or *Tau* coefficient (Saaty [32], Kendall [18]). An approximate measure of the consistency of the expert may be a pattern of "fuzzy cycles" in the matrix of comparisons. Let A be a comparison matrix where a_{ij} is expressed as a linguistic term. If we take any three compared objects (ijk), a cycle occurs where i is preferred over j AND j is preferred over k AND k is also preferred over i. If the preferences of the proposed approach have certain levels of intensity, it is reasonable to measure the "strength" of the cycle as the truth value of the following pattern.

$$\mathrm{a}_{ij} \text{ is STRONG AND } \mathrm{a}_{jk} \text{ is STRONG AND } \mathrm{a}_{ki} \text{ is STRONG} \tag{11}$$

The cycles can be illustrated easily in a graph that represents the matrix of preferences (Maa et al. [23]). Inconsistencies in the proposed pattern measurement are determined by the level of truth for the assessed pairs of preferences. If this level is smaller for the pairs that generate the cycle, the relationship is less inconsistent.

Fig. 10 shows an example of a matrix that contains comparisons of three objects. The first part (Fig. 10(a)) represents the linguistic expressions while the second part (Fig. 10(b)) shows the truth values after applying the definition of Fig. 9. The directed graph showing the relationships between the objects is provided in Fig. 10(c).

For the matrix shown in Fig. 10, pattern (11) allows us to determine the truth value of the cycle in the following manner:

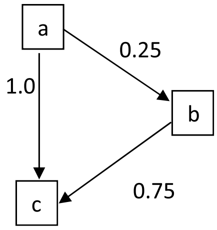
$$\text{Truth of (11)} = \min\big(\mathrm{t(a, b), t(b, c), t(c, a)}\big) = 0.$$

	a	b	c
a	x	MED	V.BIG
b	x	x	BIG
c	x	x	x

(a)

	a	b	c
a	x	0.25	1.0
b	x	x	0.75
c	x	x	x

(b)



(c)

Fig. 10. Example showing a comparison matrix, truth values, and a digraph representing the matrix relationships.

	a	b	c
a	x	MED	x
b	x	x	BIG
c	V.BIG	x	x

	a	b	c
a	x	0.25	x
b	x	x	0.75
c	1.0	x	x

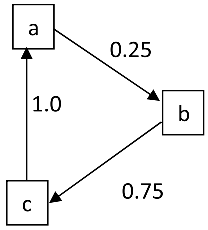


Fig. 11. The matrix from Fig. 10 with a switched a–c relationship.

We obtain a cycle by changing the direction of the a–c relationship in Fig. 10. This situation is depicted in Fig. 11.

For the matrix shown in Fig. 11, pattern (11) allows us to determine the value of the cycle as follows:

$$\text{Truth of (11)} = \min\big(\text{t(a, b)}, \text{t(b, c)}, \text{t(c, a)}\big) = 0.25.$$

As in the previous discussion, t (..) denotes the "value of truth".

Finding all of the cycles in the comparisons matrix is not easy if the number of comparisons is high. Thus, it is useful to propose a specific method for calculating (assessing) the "truth value" of pattern (11), which adequately represents the inconsistency across the array. Let A1 be a matrix of truth values t(l), which is obtained by calculating the truth values for the expert assessments according to the definitions of the patterns and variables. In addition, zeros are entered in the matrix in all of the cells that do not represent positive preferences and on the diagonal. As shown in the examples above, the matrices can be represented by a digraph G, where the vertices represent the objects being compared and the edges (directed in one way) are the levels of the "preference truth values". In this digraph, a path is a sequence of vertices, e.g., from i to k, and the path length is the number of edges connecting the nodes between i and k.

Let $A^{(k)}$ be the k-th order max–min matrix product (i.e., the k-th power), where max and min operations are used instead of $+$ and x, respectively. It is easy to see that if we calculate the third power of the matrix, in the sense of the max–min product on the diagonal of the matrix obtained, we will determine the maximum truth value of pattern (11) for all cycles of length 3 that involve a specific element (object) in the matrix. The average of these values appears to be a good way to estimate the consistency (inconsistency) of the matrix, and thus the ratings (or rather expert) quality. A simple numerical and graphical illustration clarifies the ideas above (Fig. 12).

By calculating the second and third max–min $\text{A}^{(1)}$ powers, we obtain Fig. 13.

The truth value of pattern (11) can now be assessed as the mean of the diagonal values, i.e., $(0.5 + 0.5 + 0.3 + 0.5)/4 = 0.45$.

In this example, the "expert" was rather inconsistent, which is expressed by the values on the diagonal of the matrix. Clearly, the randomly assigned values in this example do not tell us much about the properties of the proposed index. In particular, if we look carefully at the example it may be noted that the value of cycle 0.5, which occurs on the diagonal for elements a, b, and d, occurs because these objects belong to the same cycle. However, it seems that these situations are less likely in a matrix that is a realistic size (usually larger). In order to investigate the properties of the proposed approach, more systematic simulation experiments were designed and conducted. The aim of this study was to analyze the relationship between the level of the matrix inconsistency and the pattern (10) truth values for hierarchies obtained using classical algorithms. In addition, these studies also focused on obtaining a distribution

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	a	b	c	d
a	0	0.5	0.3	0
b	0	0	0	0.5
c	0	1.0	0	0.8
d	0.7	0	0	0

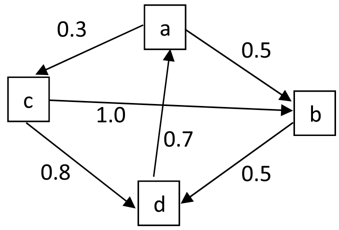

Fig. 12. Example of matrix $A^{(1)}$ and the digraph representation.

	0.5	0	0	0.3
	0	0.5	0.3	0
$A^{(3)}$	0.5	0.5	0.3	0
	0	0.3	0	0.5

Fig. 13. Third power of the example matrix $A^{(1)}$.

of pattern truth values for various parameters based on the comparisons matrix. These data can be used to determine the limits of the expert consistency in the proposed approach, analogous to Saaty's CR indicator. To perform the experiments, a computer program was developed to implement the ideas discussed and to use the classical algorithm "insert" to search for the optimal hierarchy, as described in detail by Marti and Reinelt [24]. The next section presents the details of this concept and the results of the simulation studies.

5. Simulation studies

Two series of simulation experiments were conducted in this study. The first aimed to find the relationships between the expert consistency level and the hierarchy obtained by the classical "insert" algorithm. The second study aimed to characterize the sensitivity of the proposed approach to possible expert errors during the comparison process.

5.1. Consistency of the comparison ratings matrix vs. the "quality" of the hierarchy obtained

The first series of experiments investigated the possible relationships between patterns (10) and (11) in terms of their truth values in different settings. These values should be intercorrelated if they represent the matrix inconsistency and the hierarchy quality, respectively. Our program implemented the classical "insert" algorithm (Marti and Reinelt [12]). The general idea of this algorithm is the systematic insertion of a selected object between two others (in the permutation) if this yields a better hierarchy in the sense of the pattern (10) truth value. Two parameters that characterized the problem were treated as independent variables. The first was the size of the comparisons matrix. Three different sizes were tested. According to Gass and Standard [13], matrix sizes of five, seven, and nine elements are used most commonly in the real problems studied with the AHP method. The second independent variable was the granularity of the rating scale, i.e., the number of linguistic terms used to assess the preference levels by experts when completing the comparisons matrix. Three, six, and nine levels are used widely in Likert-type scales. In particular, nine levels allowed comparisons with the AHP, which is the accepted standard. For simplicity, we assumed that the appropriate levels of the fuzzy definitions expressed opinions and definitions for the pattern variables, which were selected such that t(l) for each expression level varied proportionally from one level to the next.

For each of the nine different types of tasks (three matrix dimensions × three scale granularities), we generated a sequence of 300 different matrices containing random values. In each case, the truth value of pattern (10) was calculated for the optimal hierarchy obtained by the "insert" algorithm, and the value of pattern (11) was determined. For each configuration type, the values of the correlation coefficients between the vectors containing the truth values of patterns (10) and (11) were calculated.

The basic results of the experiments are shown in Table 1 and Figs. 14 and 15.

Table 1
Correlation coefficients between the values of the estimated truths for patterns (10) and (11) after using the "insert" algorithm in simulation experiments with matrices of different sizes and granularities to obtain the comparisons scale.

	$N = 5$	$N = 7$	$N = 9$
Granularity = 3	−0.69	−0.62	−0.56
Granularity = 6	−0.67	−0.57	−0.58
Granularity = 9	−0.63	−0.57	−0.57

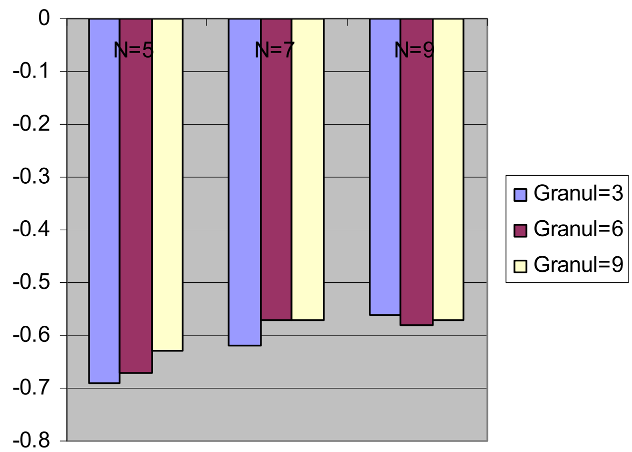

Fig. 14. Correlation coefficients for the truth values of patterns (10) and (11), which were obtained in 300 simulation experiments with different granularities and matrix sizes.

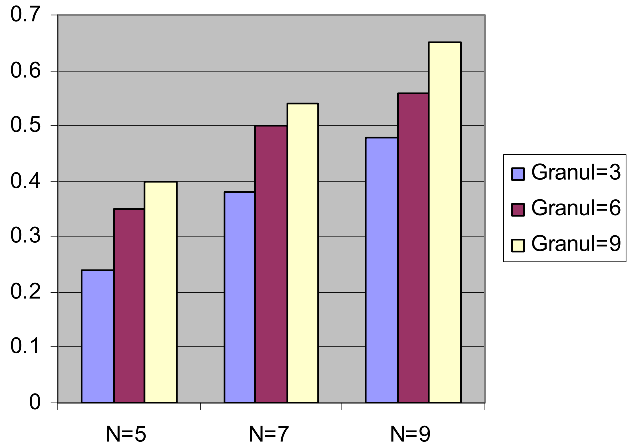

Fig. 15. Results from Table 2, i.e., the truth values for pattern (11) assessed based on the means of the appropriate matrix diagonals.

The main conclusion based on Table 1 is that the correlation between the two factors analyzed was quite high; thus, pattern (11) can be treated as a specific expert consistency factor to some extent. The correlations were rather stable (excluding the matrixes for $N = 5$) and almost independent of the granularity size (the number of linguistic expressions are used by the expert). In general, the correlations were lower for the larger matrices, as shown in Fig. 14.

To use the estimated truth value of pattern (11) by calculating the average of the diagonal elements of the matrix A raised to a power of 3 (max–min product) according to a convention similar to the CR rate, this value should be related to the mean value obtained for the set of randomly filled matrices. The simulation results in terms of the mean values (obtained for 300 randomly generated matrices in each case) are summarized in Table 2 and Fig. 14.

As shown in Fig. 14, the proposed measure of inconsistency (the mean of the diagonal elements) depends strongly on the matrix size and the granularity of the measurement scales. The random values in Table 2 should be considered

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Table 2
Truth values calculated as the average of the diagonal elements for randomly generated matrices.

	$N = 5$	$N = 7$	$N = 9$
Granul = 3	0.24	0.38	0.48
Granul = 6	0.35	0.50	0.56
Granul = 9	0.40	0.54	0.65

when the consistency of a real comparisons matrix is assessed. Naturally, the results presented have a rather limited scope for applications to the assumed linearity of truth value changes. It is possible that different assumptions about the representation of linguistic expressions (fuzzy set definitions) might change the relationships discussed. Based on Saaty's CR indicator, we may assume that an expert who achieves a truth value (11) of less than 0.1, where the appropriate truth value is computed for a randomly generated matrix (Table 2), can be considered consistent. However, it should be noted that the proposed CR threshold value is debatable. The possibility of obtaining high consistency in terms of the CR value depends greatly on many factors. For example, Grobelny and Michalski [17] and Michalski [25] showed that the gender of subjects had a significant effect on the CRs. These studies showed that females were generally less consistent than male participants. Furthermore, the experimental context may affect the consistency of the comparisons considerably. For example, the mean CR values regarding various graphical conditions [16,22,23] exhibit great differences. Thus, the overall average CR in [17] was almost twice the mean CR in [26]. The results in [26] also show that even small changes in the experimental context may affect the consistency because the same participants were markedly less consistent before performing efficiency testing compared with that after executing the experimental task.

5.2. *Sensitivity analysis*

The basic premise of building soft, approximate decision support tools is to match appropriate tools to the nature of the decisions made by experts. It appears that the use of imprecise (fuzzy) categories similar to language expressions is more similar to the natural treatment of real data by a human expert. This approach is less precise and it yields approximate assessments, where the hierarchy constructed is not very precise. Thus, by using vague and approximate evaluations, we can expect to obtain multiple hierarchies that are "indistinguishable" for an expert. Therefore, it is natural to expect that soft methods should also be more robust to errors in the assessment process. In order to verify these properties, we performed a series of simulations to compare the sensitivity of the proposed approach with the classical AHP method. This method was based on the experimental work of Gass and Standard [4], who analyzed the sensitivity of the AHP using actual hierarchy ratings. The main sensitivity index was the number of changes in the first and/or second positions of the vector obtained (using AHP) after changes in the pairwise comparisons matrix of elements. In their study, changes were made to each value in the matrix of pairwise comparisons by moving one category up or down (with a probability of 50%). A standard nine-point rating scale for the intensity of preference was used for each evaluation. The simulations were performed 50 times for each of 60 real-life problem matrices with 5–9 alternatives. The sensitivity was evaluated by counting the number of vectors where a change (in the first and/or second position in the hierarchy obtained by AHP) occurred in the relationship relative to the AHP hierarchy derived from the original data.

The experiment was designed to compare the sensitivity of the pattern-based assessment with the data obtained by Gass and Standard to some degree [13]. The second natural benchmark in the simulation experiments was fuzzy AHP, which was suggested by an anonymous reviewer. The following procedure was followed.

(A) For a given granularity (the number of preference levels used by an expert) and a given dimension N, the matrix A_{ij} containing truth values t(l) was generated randomly with a linear distance between the successive "degrees of truth."

(B) The "optimal" vector of preferences was found for this matrix using the "insert" algorithm.

(C) Each of the generated matrix truth values was moved up or down one level (with a probability of 50%), as described by Gass and Standard [13]. The preference vector was determined again and compared with the first vector in terms of the elements in the first and/or second positions. This procedure was repeated 300 times in each case for the matrix dimension and the granularity of the expert opinions. Three granularities (three, six, and nine) and two matrix sizes (five and nine, i.e., the extreme sizes used by Gass and Standard [13]) were tested.

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Table 3
Membership functions for the linguistic scale used in the sensitivity analysis.

Fuzzy number	Linguistic	Scale of fuzzy number
9	Perfect	(8, 9, 10)
8	Absolute	(7, 8, 9)
7	Very good	(6, 7, 8)
6	Fairly good	(5, 6, 7)
5	Good	(4, 5, 6)
4	Preferable	(3, 4, 5)
3	Not bad	(2, 3, 4)
2	Weak	(1, 2, 3)
1	Equal	(1, 1, 1)

Table 4
Percentage of changes in the first and/or second positions in the vectors hierarchy based on 300 randomly generated matrices compared with the experimental results obtained by Gaas and Standard using the AHP [13].

	Per cent of a change $N = 5$	Per cent of a change $N = 9$
Granularity = 3	23	47
Granularity = 6	17	29
Granularity = 9	7	15
Fuzzy AHP	36	60
AHP (G & S)	54.5	100

The sensitivity of fuzzy AHP was examined in a similar manner, where we implemented fuzzy AHP as proposed by Buckley and Uppuluri [3]. In this method, the preference vector is determined by computing the geometrical means of the fuzzy values for the rows of the comparisons matrix and further defuzzifying the fuzzy numbers obtained. Similar to Sun [38], we considered nine triangular fuzzy set definitions for linguistic expressions, which corresponded to Saaty's classical AHP. The details are given in Table 3.

The experimental procedure was analogous to that applied in the first part of the experiment and it comprised the following steps.

(A) For a given dimension N (5 or 9), the matrix A_{ij} that contained the fuzzy sets defined in Table 3 and their fuzzy reciprocals was generated randomly.

(B) The resulting preference vector was found according to the geometric means of the fuzzy values in each row. The numbers were then defuzzified by calculating the arithmetic means of the lower, middle, and upper values of the triangular numbers.

(C) Each of the matrix fuzzy numbers generated was moved up or down one level (with a probability of 50%), as described by Gass and Standard [13]. The preference vector was then determined again for the modified matrix and compared with the original with respect to the first and/or second positions in the hierarchies obtained.This procedure was repeated 300 times for each matrix dimension. The results of these experiments are shown in Table 4.

As shown in Table 4, the highest percentage change occurred with a small granularity size and a high matrix dimension (N). This is intuitive because a change in one category in this case represented the largest relative change in the assessment of preferences. However, the most interesting finding is that in the AHP experiment [13] with nine elements in the comparisons scale, the percentage change was significantly higher (54.5% for five element matrix and 100% for $N = 9$). Thus, the sensitivity of the implemented version of fuzzy AHP was between the proposed method and the classical AHP

6. Conclusions and summary

The proposed approach based on LPs appears to be promising for analyses based on expert pairwise comparisons due to some of its characteristics. First, the experiments performed in this study showed that an expert's consistency can be evaluated in a simple manner using the LP (11). The mean truth values that satisfied this pattern correlated with the corresponding mean truth values calculated for pattern (10). However, to apply them in practice, additional

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statistical analyses will be required in different contexts, e.g., different values of N and different granularities in the assessment scale, as well as various linguistic expression fuzzy representations. Second, the LP approach is very flexible. This flexibility is particularly applicable to the possible definitions of variables and the fuzzy representation of a linguistic relationship. Using these variables and relationships, we can encode common sense and imprecise knowledge about the phenomena being studied, as well as more advanced scientific information. Third, as demonstrated by the simulation experiments, the proposed approach is less precise but also less susceptible to human expert errors. The results suggest that many of the hierarchy vectors can be evaluated as indistinguishable, which means that the same hierarchy can be obtained from slightly different matrix evaluations. It appears that this situation is more consistent with the natural way people perform evaluations compared with the strict and highly sensitive AHP approach. This higher error tolerance and flexibility when defining patterns and fuzzy representations for expert opinions using the proposed method compared with the version of fuzzy AHP investigated would probably be similar with other fuzzy-based approaches.

This study did not address the problem of searching for an optimal hierarchy for a given comparisons matrix. In addition, only one of the classical algorithms was tested in the simulation experiments. It should be noted that Grobelny [16] formulated the theorem of "maximal truth", which allows the calculation of the upper bound of the maximum mean truth for the patterns (10 or 11) of a given matrix A_{ij}. It is possible that this procedure may be appropriate for constructing an analytical method to search for hierarchies using the "branch and bound" technique. This approach could be more beneficial in practice (as shown by Gass and Standard [13], among others) because the size of the typical hierarchies explored in pairwise comparisons of objects rarely exceeds nine elements.

An interesting direction for future research based on the proposed method may be studying the effects of applying various techniques to the determination of truth values for the left-hand and/or right-hand sides of patterns based on the sensitivity of the results and their general quality. In particular, it may be useful to examine how assumption of more or less optimistic decision-makers (the use of possibility and necessity measures) might influence the relative truth values in various application contexts. It may also be interesting to use more than one LP in a multi-aspect analysis of object hierarchies.

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References

[1] B. Bouchon-Meunier, D. Dubois, L. Godo, H. Prade, Theory in approximate reasoning and plausible reasoning, in: J.C. Bezdek, et al. (Eds.), Fuzzy Sets in Approximate Reasoning and Information Systems, Kluwer Academic Publishers, 1999.

[2] J.J. Buckley, Fuzzy hierarchical analysis, Fuzzy Sets Syst. 17 (1985) 233–247.

[3] J.J. Buckley, V.R.R. Uppuluri, Fuzzy hierarchical analysis, in: V.T. Covello, L.B. Lave, A. Moghissi, V.R.R. Uppuluri (Eds.), Uncertainty and Risk Assessment, Risk Management and Decision Making, Plenum Press, New York, NY, 1984, pp. 389–401.

[4] J.J. Buckley, T. Feuring, Y. Hayashi, Fuzzy hierarchical analysis revisited, Eur. J. Oper. Res. 129 (2001) 48–64.

[5] F.J. Cabrerizoa, R. Ureñab, W. Pedrycz, E. Herrera-Viedmab, Building consensus in group decisionmaking with an allocation of information granularity, Fuzzy Sets Syst. 255 (2014) 115–127.

[6] A.S. Carrie, The layout of multi-product lines, Int. J. Prod. Res. 13 (1975) 541–557.

[7] M. Cayrol, H. Farreny, H. Prade, Fuzzy pattern matching, Kybernetes 11 (1982) 103–116.

[8] I. Chamodrakas, D. Batis, D. Martakos, Supplier selection in electronic marketplaces using satisficing and fuzzy AHP, Expert Syst. Appl. 37 (2010) 490–498.

[9] S. Chanas, P. Kobylanski, A new heuristic algorithm solving the linear ordering problem, Comput. Optim. Appl. 6 (1996) 191–205.

[10] C.-H. Cheng, D.-L. Mon, Evaluating weapon system by analytical hierarchy process based on fuzzy scales, Fuzzy Sets Syst. 63 (1994) 1–10.

[11] M. Davies, Adaptive AHP: a review of marketing applications with extensions, Eur. J. Mark. 35 (2001) 872–893.

[12] H. Deng, Multicriteria analysis with fuzzy pairwise comparison, Int. J. Approx. Reason. 21 (1999) 215–231.

[13] S.I. Gass, S.M. Standard, Characteristics of positive reciprocal matrices in the analytic hierarchy process, J. Oper. Res. Soc. 53 (2002) 1385–1389.

[14] O. Gogus, T.O. Boucher, Strong transitivity, rationality and weak monotonicity in fuzzy pairwise comparisons, Fuzzy Sets Syst. 94 (1998) 133–144.

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[15] J. Grobelny, The 'linguistic pattern' method for a workstation layout design, Int. J. Prod. Res. 26 (1988) 1779–1798.
[16] J. Grobelny, The fuzzy approach to facilities layout problems, Fuzzy Sets Syst. 23 (1987) 175–190.
[17] J. Grobelny, R. Michalski, Various approaches to a human preference analysis in a digital signage display design, Hum. Factors Ergon. Manuf. 21 (2011) 529–542, http://dx.doi.org/10.1002/hfm.20295.
[18] M.G. Kendall, Rank Correlation Methods, 3rd edn., Charles Griffin, London, 1962.
[19] W. Koczkodaj, Testing the accuracy enhancement of pairwise comparisons by a Monte Carlo experiment, J. Stat. Plan. Inference 69 (1) (1998) 21–31.
[20] F. Kong, H. Liu, Applying fuzzy analytic hierarchy process to evaluate success factors of e-commerce, Int. J. Inf. Syst. Sci. 1 (2005) 406–412.
[21] J.B. Kruskal, Nonmetric multidimensional scaling: a numerical method, Psychometrika 29 (1964) 115–129.
[22] P.J.M. Laarkoven, W. Pedrycz, A fuzzy extension of Saaty's priority theory, Fuzzy Sets Syst. 11 (1983) 229–241.
[23] J. Maa, Z. Fanb, Y.P. Jiangb, J.Y. Maoc, L. Maa, A method for repairing the inconsistency of fuzzy preference relations, Fuzzy Sets Syst. 157 (2006) 20–33.
[24] R. Martí, G. Reinelt, The Linear Ordering Problem – Exact and Heuristic Methods in Combinatorial Optimization, Springer, 2011.
[25] R. Michalski, Examining users preferences towards vertical graphical toolbars in simple search and point tasks, Comput. Hum. Behav. 27 (2011) 2308–2321, http://dx.doi.org/10.1016/j.chb.2011.07.010.
[26] R. Michalski, The influence of color grouping on users' visual search behavior and preferences, Displays 35 (2014) 176–195, http://dx.doi.org/10.1016/j.displa.2014.05.007.
[27] L. Mikhailov, Deriving priorities from fuzzy pairwise comparison judgements, Fuzzy Sets Syst. 134 (2002) 365–385.
[28] R. Muther, Systematic Layout Planning, 2nd ed., John Wiley & Sons, New York, 1996.
[29] A. Pollitt, Comparative judgement for assessment, Int. J. Technol. Des. Educ. 22 (2012) 157–170.
[30] H. Prade, A computational approach to approximate and plausible reasoning with applications to expert systems, IEEE Trans. Pattern Anal. Mach. Intell. 3 (1985) 260–283.
[31] R.W. Proctor, T. Van Zandt, Human Factors in Simple and Complex Systems, Allyn and Bacon, Boston, 1994.
[32] T.L. Saaty, A scaling method for priorities in hierarchical structures, J. Math. Psychol. 15 (1977) 234–281.
[33] T.L. Saaty, The Analytic Hierarchy Process, McGraw-Hill, New York, 1980.
[34] T.L. Saaty, The Analytic Hierarchy Process, RWS Publications, Pittsburgh, 1996.
[35] J.M. Scher, A generalized Thurstonian paired comparison multicriteria heuristic model for peer evaluation of individual performance on IS team projects, IS Ed. Journal 8 (2010) 1–9.
[36] S.S. Stevens, On the psychophysical law, Psychol. Rev. 64 (1957) 153–181.
[37] N. Subramanian, R. Ramanathan, A review of applications of analytic hierarchy process in operations management, Int. J. Prod. Econ. 138 (2012) 215–241.
[38] C.-C. Sun, A performance evaluation model by integrating fuzzy AHP and fuzzy TOPSIS methods, Expert Syst. Appl. 37 (2010) 7745–7754.
[39] L.L. Thurstone, A law of comparative judgment, Psychol. Rev. 34 (1927) 273–286.
[40] H. de Vries, Finding a dominance order most consistent with a linear hierarchy: a new procedure and review, Anim. Behav. 55 (1998) 827–843.
[41] X. Wang, An investigation into relations between some transitivity related concepts, Fuzzy Sets Syst. 89 (1995) 257–262.
[42] M. Xiaa, J. Chen, Consistency and consensus improving methods for pairwise comparison matrices based on Abelian linearly ordered group, Fuzzy Sets Syst. 266 (2015) 1–32.
[43] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.
[44] L.A. Zadeh, PRUF – a meaning representation language for natural languages, Int. J. Man-Mach. Stud. 10 (1978) 395–460.
[45] H.J. Zimmermann, Fuzzy Sets Theory and Its Applications, fourth edition, Kluwer Academic Publisher, 2001.

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