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THE FUZZY APPROACH TO FACILITIES LAYOUT PROBLEMS

Jerzy GROBELNY
Institute of Production Engineering and Management Technical University of Wrocław, Poland

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Facilities layout problems are discussed briefly. Some 'classical' approaches to the solution of these problems are described. The fuzzy approach is presented. This approach is based on Zadeh's possibility theory and Łukasiewicz's multivalued implication formula. The heuristic algorithm for solving the fuzzy facilities layout problem is proposed and exemplified by a simple numerical example.

Keywords: Facilities layout, Possibility theory, Multivalued logic.

1. Introduction

The facilities layout problem (FLP) has been the subject of interest of specialists in Management Science [3] and the ergonomics field [1] since the early sixties.

Quite little attention in the hitherto known investigations has been paid to the problems of gathering the input data for algorithms. Seeberger and Wierwille [9] developed a statistical method of determining the 'traffic intensity' degree between instruments (link values) at the work places for ergonomic needs, whereas as early as in 1966 Gavett and Plyter [2] noticed the fact that obtaining reliable data for the FLP problem is often very hard.

In the present work a simple formal formulation is being proposed for the situation where (for various reasons) it is impossible to obtain accurate numerical estimations of the 'link values' (and significance, convenience etc.) of the facilities and location places, but it is possible to gain approximated estimations represented by means of fuzzy sets. A heuristic algorithm has been suggested to solve FLP under the conditions of non-precisely formulated data. The algorithm is based on the modification of a widely known approach HC-66 (Hillier and Connors [5]). Applying the estimates similarly to the natural language the suggested approach may be one more step in the direction of layout algorithms regarding the human designer intuition as postulated by Bonney and Williams [1]. This direction seems right especially in the light of investigation results obtained by Scriabin and Vergin [8] who pointed that man and his intuition are more advantageous than some heuristic algorithms serving to solve layout problems.

2. Facilities layout problem

For the convenience of the reader we present here a description of the facilities layout problem (FLP). We will characterize some of the most often used notions and definitions of variables used in two main application areas of FLP – ergonomics and production engineering.

The FLP can be defined as the following task:

Locate n facilities at n fixed locations (of the set of m locations, $m \geqslant n$) in such a way to maximize (or minimize) a certain objective criterion function, with the constraint that one and only one facility may be assigned to each location.

In the field of ergonomic design, controls and displays constituting a given workstation are considered as 'facilities' [1]. Production engineers usually define facilities as indivisible workstations, machines, pieces of equipment or separate departments.

Location places (in both considered areas) can be physically determined (e.g. by technology) or one can assume that the whole space being at one's disposal can be divided into elementary subspaces called locations.

There are two distinct types of criteria that can be used to optimize the facilities layout: first order criteria and second order criteria [9]. First order criteria describe the relationship between each given facility and its location. Second order criteria, on the other hand, describe the relationships that exist among pairs of facilities and 'distances' between the places of their locations.

Installation costs are an example of the first order criterion used in plant layout problems [5]. Optimization in this case is to find such an arrangement of facilities which minimizes the sum of installation costs over all facilities.

The minimization of the sum of products of the 'importance degree' times 'distance from designed panel center' is an example of the first order criterion used in an ergonomic panel design. The 'importance', is usually defined by an expert as a numerical weight which represents the expert's preference based on the role played by a given facility in a designed workstation. In the more general workstation design process the 'distance from the panel center' is replaced by a 'convenience degree' of a given location, usually determined by an expert (ergonomist) on the basis of empirical knowledge of psychophysical abilities of man (e.g. field of view features, range of limbs manipulation).

A total cost of materials and goods transportation between workstations is a typical second order criterion used in production engineering problems. For one pair of fixed facilites it is generally expressed as a product of the 'traffic intensity degree' and the 'distance' between their locations [2]. The 'traffic intensity' (a kind of facilities dependence measure) is most often the rate at which materials (or goods) will be transferred between a given pair of facilities, and the 'distance' is the cost (or index of cost) of having to transfer a unit of material between the pair of locations. This cost measure may literally be a linear distance, but not necessarily so. Because of the complexity of interrelationships in actual production systems the 'traffic intensity degree' as well as the 'distance' have to be aggregate values. Experts' opinions are in this case often the only source of gathering data for designers as there are difficulties with 'physical measurement' and changes of data in the course of time. To unify in some way the 'experts based' approach special REL-charts were proposed [7]. An expert marks his opinion on each facility pair dependence (called 'closeness') in the six step closeness priority standard utilizing the vowel letters (A – absolutely necessary, E – especially important, I – important, O – ordinary closeness, U – unimportant and X – not desirable). In computational algorithms letters are replaced by numerical weights.

The second order criterion for ergonomical purposes can be stated similarly but the terms used above need to be redefined. The 'traffic intensity degree' – traditionally called 'link value' – is most often defined as a "measure of the desirability of having two instruments located near to each other to facilitate operation of the system" [2]. The 'distance' is usually physical, linear distance which the hands (or eyes) must travel to 'manipulate' two instruments. The formal statistical method, based on observations of frequency of use for determining link values was proposed by Seeberger and Wierwille [9]. Limitations of this approach (costs and similarity of designed and examined workstations) imply the use of imprecise data based on experts' opinions [1].

A logical, practical example of the FLP is the case of plant layout where there are a number of machines, pieces of equipment or departments that must be assigned to a set of locations. The locations are distance oriented and fixed in a space. There is a rate of flow of materials (parts, people) between each pair of facilities. The costs of installation for each facility in each location are determined.

One of the most frequent formal descriptions of the facilities layout problem (FLP) to be found for this case is the proposition by Hillier and Connors [5] to formulate the problem in the convention of the quadratic assignment problem (QAP). It may be presented as follows: Let

n = number of laid-out (indivisible) facilities and locations.
c_{ij} = cost (per time unit) of the assignment of facility 'i' to place 'j',
d_{jr} = unitary cost of the 'passage' from location point 'j' to location point 'r',
f_{ik} = number of passages between 'i' and 'k' facilities

$$a_{ijkr} = \begin{cases} f_{ik} d_{jr} & \text{if } i \neq k \text{ or } j \neq r, \\ c_{ij} & \text{if } i = k \text{ and } j = r, \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if the facility '}i\text{' is assigned to place '}j\text{'}, \\ 0 & \text{otherwise.} \end{cases}$$

Now the problem of facilities layout planning may be presented as

$$\text{Minimize} \quad \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{r=1}^{n} a_{ijkr} x_{ij} x_{kr} \tag{1}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n, \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n, \tag{3}$$

$$x_{ij} = 0 \text{ or } 1, \quad i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n. \tag{4}$$

In such a model it is especially difficult to determine c_{ij} costs and f_{ik}-number as precisely specified data.

3. The fuzzy approach

In the face of the situation, particularly of frequent difficulties in precise determination of data concerning facilities interrelationships as well as not too precise approximate algorithms of the facilities layout, it seems interesting to alternately formulate the FLP in fuzzy terms and manipulate these categories in accordance with the basic principles of the fuzzy sets theory. Such an approach will make it possible to formulate the facilities layout problem in a way closer to the human designer intuition than in the case of any other formal 'classical' conception. Intuitions of the known approaches to the solving of the problem under discussion as well as to the evaluation of given layout can be limited to the following:

The facilities forming a set should be laid out so that: (a) facilities of strong mutual interrelationship, generally called 'link values' (understood as the flow of materials, energy, information, etc.) are situated close to each other, and (b) facilities in a sense 'important' should be located at in a sense 'convenient' locations. Case (b) occurs mainly in ergonomic problems and 'importance' is most often defined as an aggregate regarding the significance of the facility for the processes taking place as a whole at the workplace being designed and its frequency of use (Seeberger and Wierwille [9]), whereas 'convenience' of the location is most often interpreted as a general degree of ease of use in relation to the person operating a given facility at a given place. In the (1)–(4) model the case (b) can be interpreted as follows:

(b′) A given facility should be located at a place where its installation costs are lowest (more precisely – an annual equivalent of these costs).

A bit more 'formally', procedures (a), (b) and (b′) can be rewritten as:

(A) IF THE LINK VALUE (of two given facilities) = 'VERY BIG' THEN THE DISTANCE (between their location places) = 'VERY SMALL'. (5)

(B) IF THE IMPORTANCE DEGREE (of a given device) = 'VERY BIG' THEN THE CONVENIENCE DEGREE (of its location place) = 'VERY BIG'. (6)

(B′) LAYOUT COST (of a given device at a given place) = 'MINIMAL'. (7)

The above statements can be regarded as 'reference propositions' or criteria according to which the given solution of the FLP can be evaluated. Of course, the expressions in quotes (in (5)–(7)) may be more complicated in specially formulated problems but most important is that they can be interpreted in general in the categories of fuzzy sets in appropriately defined spaces.

Let L_{ij} be a fuzzy set in the space X, determining the link value of 'i' and 'j' facilities of the membership function $L_{ij}(x)$, whereas D_{kr} is a fuzzy set in the space Y, determining the distance between 'k' and 'r' places of the membership function $D_{kr}(y)$. The spaces X and Y can be defined, taking into account the way of data gathering generally as (a) spaces of discourse of physical dimension (operationally defined), for example material flows in kg/h (as link values) or cost per transferred unit in \$ (as a 'distance'), (b) artificial space of linguistic terms representation – for example the interval of the real line to represent linguistic expressions of the type 'BIG', 'SMALL', 'MEDIUM' etc. for the level of 'link values'.

With these assumptions it is possible, making use of the methodology of Zadeh [12], to calculate the truth value of L_{ij} in reference to the proposition – criterion from the expression (A) (LINK VALUE = 'VERY BIG') as:

$$p_1^{(ij)} = \mathrm{POSS}(L_{ij} \text{ is V.BIG}) = \sup_{x \in X} \{L_{ij}(x) \wedge \text{V.BIG}(x)\} \tag{8}$$

where V.BIG(x) is the membership function of the expression VERY BIG in the space X and '$\wedge$' the minimum operation. By analogy,

$$q_1^{(kr)} = \mathrm{POSS}(D_{kr} \text{ is V.SMALL}) = \sup_{y \in Y} \{D_{kr}(y) \wedge \text{V.SMALL}(y)\} \tag{9}$$

where V.SMALL(y) is the membership function of the expression 'VERY SMALL' in the space Y and $q_1^{(kr)}$ the truth value of D_{kr} in relation to the proposition DISTANCE = 'VERY SMALL'. Here, we want to point out that the way of constructing fuzzy sets representing linguistic expressions in (8) and (9), i.e. 'reference propositions', and also the question of a linguistically formulated input data (L_{ij}, D_{kr}) representation in appropriate spaces, seems to be essential. Some rationales for these problems were formulated by Zadeh [12]. Some attention to many practical problems of that kind was paid in works of Freksa [4] and Yager [11]. Because, after all, problems under discussion are complex, and 'separate' questions arise, we shall not pay more attention to them, all the more since the course of our concepts is 'independent' of the questions mentioned above. The estimation of the truth value of the implication $L_{ij} \Rightarrow D_{kr}$ (in relation to the criterion (A)) may be performed according to the multi-modal logical formula by Łukasiewicz (Tsukamoto [10]) as:

$$O_{ij}^{kr} = \min\{1,\ 1 - p_1^{(ij)} + q_1^{(kr)}\} \tag{10}$$

where O_{ij}^{kr} can be interpreted as a satisfaction degree of the criterion (A) if the facilities 'i' and 'j' are laid out at 'k' and 'r' places respectively. Formula (10) describes the truth value in a generalized implication. Let a denote the truth value of an expression A and b the truth value of an expression B. Let $I(a, b)$ denote the truth value of implication IF 'A' THEN 'B'; then one can prove that $I(a, b)$ defined as in (10) (i.e. $I(a, b) = \min\{1, 1 - a + b\}$) has the following 'reasonable' properties:

$$\begin{array}{ll} 1^\circ & I(a, 1) = 1, \\ 2^\circ & I(0, a) = 1, \\ 3^\circ & I(1, a) = a, \\ 4^\circ & I(a, b) \geq I(a, c) \text{ iff } b \geq c, \\ 5^\circ & I(a, b) \geq I(c, b) \text{ iff } a \leq c. \end{array} \tag{11}$$

Of course another function fulfilling the above can replace Łukasiewicz's formula in (10).

A similar approach can be proposed for the estimation of a given facility-location system according to the criterion (B).

Let I_i, denoting an importance degree of the 'i' facility, be a fuzzy set in Z of the membership function $I_i(z)$, and C_k, denoting the convenience degree of place k, be a set in V of the membership function $C_k(v)$; then

$$p_2^{(i)} = \mathrm{POSS}(I_i \text{ is } \mathrm{V.BIG}) = \sup_{z\in Z}\{I_i(z) \wedge \mathrm{V.BIG}(z)\} \tag{12}$$

where $p_2^{(i)}$ is the truth value of I_i in relation to the proposition criterion (B) (i.e. 'VERY BIG'), whereas V.BIG(z) is the membership function of the fuzzy set representing the expression VERY BIG in the space Z and

$$q_2^{(k)} = \mathrm{POSS}(C_k \text{ is } \mathrm{V.BIG}) = \sup_{v\in V}\{C_k(v) \wedge \mathrm{V.BIG}(V)\} \tag{13}$$

where $q_2^{(k)}$ denotes the truth value of C_k in relation to the expression VERY BIG', and VERY BIG(v) is the membership function of the fuzzy set corresponding to the notion 'VERY BIG' in the space V.[1]

When Q_i^k denotes the satisfaction degree of the criterion (B) through the location of the i-th facility at the k-th place, one can determine

$$Q_i^k = \min\{1, 1 - p_2^{(i)} + q_2^{(k)}\} \tag{14}$$

and in an alternative case of the criterion (B′),

$$Q_i{}^k = \mathrm{POSS}(K_{ik} \text{ is } \mathrm{MINIMAL}) = \sup_{c\in C}\{K_{ik}(c) \wedge \mathrm{MINI}(c)\} \tag{15}$$

where K_{ik} is the cost of the installation of facility 'i' at place, 'k', and $K_{ik}(c)$ and MINI(c) are the membership functions of the fuzzy sets representing installation cost K_{ik} and 'minimal' cost in the space C, respectively.

Formula (10) enables us to calculate the truth value of the criterion (A) being satisfied for the pair of i-j facilities layed out at k-r places. Assuming, as in the (1)–(4) model, that the number of location places equals that of the layed out facilities and amounts to n, a general evaluation of a given layout system $p = (p_1, \ldots, p_n)$, in which p_i is the number of locations to which the i-th facility is assigned, in relation to the criterion (A) can be formulated as follows:

$$O(p) = \frac{2}{n^2 - n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} O_{ij}^{p_ip_j}. \tag{16}$$

Similarly, for the (B) and (B′) criteria,

$$Q(p) = \frac{1}{n}\sum_{i=1}^{n} Q_i^{p_i}. \tag{17}$$

[1] We want to point out that functions of the type V.BIG(x) can be treated as standards (Zadeh [12]) or can be defined individually because of many reasons. But either in the first or the second case, generally the spaces X, Y, Z and V are different. The spaces Z and V can be determined in the ways discussed in relation to X and Y.

Estimations $O(p)$ and $Q(p)$ represent 'the mean truth value' per one pair of the facilities and one facility, respectively; it results from satisfying the requirements, formulated in (A) and (B) (or (B')) criteria, in a given layout characterized by permutation p.

A general estimation (in respect of both criteria jointly) must be a result of the aggregation of O and Q estimates. It seems that in this case (of operating the truth value estimates) there is much to be said for performing this by means of the function of t-norm properties, i.e. the function $T:[0, 1]^2 \to [0, 1]$ for which:

1° $T(a, 1) = a$,

2° $T(a, b) = T(b, a)$,

3° $T(a, b) \geqslant T(c, d)$ iff $a \geqslant c$ and $b \geqslant d$,

with an additional condition

3°′ $T(a, b) > T(a, d)$ iff $b > d$ and $a > 0$,

which correspond to the intuition that the increase of truth in one component makes a general estimate increase.

The operation of multiplication is the one to satisfy these properties. It can be generally written

$$G(p) = T^{+}(O(p), Q(p)) \tag{18}$$

where T^{+} denotes a function satisfying the properties 1°–3°.

One can easily notice that in this conception the formulated criteria of the optimum layout of the facilities in a given space enable the estimation not only in 'pure' fuzzy problems but also in 'mixed ones' (e.g. the criterion of distance being determined as a range or fuzzy set and the distance between places (D_{kr}) being accurately measured). A particular case of the proposed conception may be the formulation of estimates and criteria as intervals. For example, criterion (A) can be represented as:

$$\text{IF LINK VALUE} = \text{`}A\text{'} \text{ THEN THE DISTANCE} = \text{`}B\text{'} \tag{19}$$

where 'A' and 'B' denote 'sharply' determined numerical intervals in corresponding spaces, then

$$p_1^{(i,j)} = \mathrm{POSS}(L_{ij} \text{ is } A) = \sup_{x \in X} \{L_{ij}(x) \wedge A(x)\} = \begin{cases} 0 & \text{if } L_{ij} \cap A = \emptyset, \\ 1 & \text{if } L_{ij} \cap A \neq \emptyset, \end{cases}$$

and

$$q_1^{(kr)} = \mathrm{POSS}(D_{kr} \text{ is } B) = \sup_{y \in Y} D_{kr}(y) B(y) = \begin{cases} 0 & \text{if } D_{kr} \cap B = \emptyset, \\ 1 & \text{if } D_{kr} \cap B \neq \emptyset. \end{cases}$$

Of course the $L_{ij}(x)$ and $A(x)$ as well as $D_{kr}(y)$ and $B(y)$ are fuzzy sets of the membership functions equal to 1 for a given interval and 0 outside it. The truth value of satisfying criterion (19) by a given layout system in this situation is calculated according to the classical implication truth value.

The hitherto presented discussion enables the evaluation of any facilities layout system and thus makes it possible to choose the best one out of many assessed ones. The problem discussed can be now formulated as the problem of discrete optimization ($\max G(p)$, where P is the set of all permutations of numbers $1, 2, \ldots, n$). Certainly for the problem dimensions $n > 10$ the review of all possible variants of the layout is ineffective even for a fast computer. Foulds [3], presenting the results of the work by Sahni and Gonzales, states that an effective, analytic algorithm for the problem (1)–(4) of the dimensions $n > 10$ is unlikely to be found (it is an NP-complete problem). Hence the attempts to find approximate methods are of great importance. Many of these heuristic methods are discussed by Foulds [3] and Bonney and Williams [1]. In the following section the heuristic algorithm, based on the HC-66 algorithm concept (Hillier and Connors [5]) is presented for solving the 'fuzzy' task formulated above.

This concept can be characterized briefly as follows. Based on data concerning 'link values', distances and installation costs (a_{ijkr} in model (1)–(4)), lower bounds on the criterion-function (1) increase are determined for all pairs 'not assigned-facility' – 'free location'.

In this way the matrix of assignments is constructed. This matrix is the basis for an assignment of facility to the location which gives in perspective a minimal increase of the criterion function. To achieve this the VAM method is used. This procedure is repeated until the moment in which $n-1$ facilitles are located. In the algorithm proposed below the course of this idea was used. Because, after all, variables and criteria in our approach are completely different from 'classical' ones, the way of construction of the 'matrix of assignments' was a different one too. A special 'maximal truth value' theorem had to be formulated and proved. The algorithm includes the case where (A)–(B) criteria occur (the joint occurrence of the (A)–(B′) criteria is a specific case of the (A)–(B) situation).

4. Heuristic algorithm

Assuming that in the case of the (A)–(B) criteria being applied the following parameters are given (in the form of fuzzy sets):

– facilities interrelationship degrees (link values) – matrix $L = [L_{ij}]_{n\times n}$,
– distances between places – matrix $D = [D_{kr}]_{n\times n}$,
– significance (importance) degrees of individual facilities – vector $I = [I_i]_{n\times 1}$,
– convenience degrees of each place – vector $C = [C_r]_{n\times 1}$,

and assuming that the number of facilities n equals that of possible places (which does not change the generality degree of considerations) one should follow the following algorithm:

1° Make the matrix L' with elements $L'_{ij} = p_1^{(ij)}$ (formula (8)) and the matrix D' with elements $D'_{kr} = q_1^{(kr)}$ (formula (9)) for $i = 1, \ldots, n$, $k = 1, \ldots, n$, $j = 1, \ldots, n$, $r = 1, \ldots, n$, assuming, for all $i = j$ and $k = r$, that $L'_{ij} = D'_{kr} = 1$. Substitute $RL' := L'$ and $RD' := D'$.

2° Make the vector I' with elements $I'_i = p_2^{(i)}$ (formula (12)) and vector C' with elements $C'_k = q_2^{(k)}$ (formula (13)) (for $k = i = 1, \ldots, n$) and then form the matrix F with elements $F_{ik} = Q_{ik}$ (formula (14)) for all 'i' and 'k'.

3° Sort out the elements in columns of matrices RL' and RD' in a decreasing order denoting the obtained matrices by L'' and D''.

4° Determine the matrix of assignments A ('truth losses') according to the following 'step criterion function':

$$A_{kl} = 1 - (F_{kl} \cdot \tfrac{1}{2}(a + b)) \tag{20}$$

where

$$a = \frac{1}{|\mathscr{L}|} \sum_{i \in \mathscr{L}} \{(1 - L'_{ki} + D'_{lp(i)}\} \wedge 1\}, \tag{21}$$

$$b = \frac{1}{n - |\mathscr{L}|} \sum_{j=1}^{n-|\mathscr{L}|} \{(1 - L''_{jk} + D''_{jl}) \wedge 1\} \tag{22}$$

in which $|\mathscr{L}|$ is the cardinality of a set of laid out elements, $\mathscr{L}$ a set of indices of laid out elements, $p(i)$ the index of location place of facility 'i'. (For the 'first pass' ($|\mathscr{L}| = 0$ and $\mathscr{L} = \emptyset$) a is assumed to be 0.)

5° Choose on the basis of the matrix A the assignment of facility 'k' to place 'l' so that it will provide the minimum 'truth loss', applying the VAM method (see Appendix); substitute $\mathscr{L} := \mathscr{L} \cup \{k\}$.

6° Delete the rows and the columns in the matrices RL', RD' corresponding to the chosen facility and its location place.

7° If one facility has been left out, locate it at a remaining place; otherwise realize the algorithm once more, beginning with step 3°.

In order to illustrate the algorithm a simple numerical example is presented in the following section.

It can be noticed that the concept of the algorithm is based generally on the branch and bound method. The basic difference in the presented proposition is the lack of 'returns' to a once rejected branch of the 'solution tree', and only one part of the 'step criterion function' (20) is estimated by finding its upper bound (22). The matrix A_{kl}, based on which assignments in the proposed algorithm are performed, may be interpreted as a matrix of 'truth-value' losses, associated with the satisfaction of the (A) and (B) criteria in the case of selecting a given layout. The smallest loss of truth value is achieved for the layout providing the maximum value for the expression in brackets in (20). The part denoted by a (21) enables one to determine the potential increase of the truth value per each new link of already laid out facilities, resulting from the location of facility 'k' at place 'l' (or the mean truth value of the facilities k having already been laid out).

The part given by 'b' (formula (22)) is an estimate of the upper bound of the perspective truth-value increase of the A criterion satisfaction, resulting from the location of facility 'k' at the place 'l'. One can formally rewrite

$$b = \max_{p \in F} \left[\frac{1}{n - |\mathscr{L}|} \sum_{j=1}^{n-|\mathscr{L}|} \{(1 - RL'_{kp_j} + RD'_{lj}) \wedge 1\} \right] \tag{23}$$

where F is the set of all permutations of indices of not laid out facilities, p a permutation of indices of not laid out facilities, and p_j the j-th element of permutation p.

It is possible to rewrite formula (23) in the form 'b' of formula (21) due to the application of 'the maximum truth-value theorem' presented together with its proof in Appendix 2.

5. Simple numerical example

To make the course of the presented approach more understandable let us consider a simple problem of an ergonomical design. There are 3 instruments which ought to be located at 3 free places at a given workstation. All data for the optimization are given as linguistic categories formulated by expert ergonomists. The experts arbitrarily constructed representations of 5 expressions used in their opinions and decided that criteria of the form (A) and (B) should be applied. Definitions of the expressions used are shown in Table 1.

The matrix of link values (facilities 1, 2, 3) is as follows:

$$L = \begin{pmatrix} \text{—} & \text{V.SMALL} & \text{BIG} \\ \text{V.SMALL} & \text{—} & \text{SMALL} \\ \text{BIG} & \text{SMALL} & \text{—} \end{pmatrix}.$$

The matrix of 'distances' (places 1, 2, 3) is defined in the following way:

$$D = \begin{pmatrix} \text{—} & \text{V.SMALL} & \text{MEDIUM} \\ \text{V.SMALL} & \text{—} & \text{V.SMALL} \\ \text{MEDIUM} & \text{V.SMALL} & \text{—} \end{pmatrix}.$$

The vectors of facility importance and convenience degrees of individual location places are given respectively by

$$I = (\text{V.BIG} \quad \text{SMALL} \quad \text{MEDIUM}), \qquad C = (\text{BIG} \quad \text{V.BIG} \quad \text{MEDIUM}).$$

In the first step of the proposed algorithm one should create the matrices L' and D'. for example,

$$L'_{12} = p_1^{(12)} = \text{POSS}(L_{12} \text{ is V.BIG}) = \sup_{x \in X} \{\text{V.SMALL}(x) \wedge \text{V.BIG}(x)\}$$

$$= \sup\{0.0, 0.0, 0.2, 0.0, 0.0\} = 0.2.$$

Calculating in this way one obtains (adding '1' on diagonals)

$$L' = \begin{pmatrix} 1.0 & 0.2 & 0.9 \\ 0.2 & 1.0 & 0.3 \\ 0.9 & 0.3 & 1.0 \end{pmatrix}, \qquad D' = \begin{pmatrix} 1.0 & 1.0 & 0.8 \\ 1.0 & 1.0 & 1.0 \\ 0.8 & 1.0 & 1.0 \end{pmatrix}.$$

We also make $RL' := L'$ and $RD' := D'$.

Table 1. Definitions of the applied linguistic expressions

	Representation space of the expressions $X = Y = Z = V$				
Denotations	1	2	3	4	5
VERY SMALL(x)	1.0	0.8	0.2	0.0	0.0
SMALL(x)	0.9	1.0	0.8	0.3	0.0
MEDIUM(x)	0.3	0.8	1.0	0.8	0.3
BIG(x)	0.0	0.3	0.8	1.0	0.9
VERY BIG(x)	0.0	0.0	0.2	0.8	1.0

Putting into practice step 2° of the algorithm, making use of the formulae (12) and (13), as above, and Table 1, one obtains

$$I' = (1.0 \quad 0.3 \quad 0.8), \qquad C' = (0.9 \quad 1.0 \quad 0.8).$$

(For example: element $l_2' = p_2^{(2)} = \text{POSS}(I_2 \text{ is } \text{V.BIG}) = \sup_{z \in Z} \{\text{SMALL}(z) \wedge \text{V.BIG}(z)\} = \sup\{0.0,\ 0.0,\ 0.2,\ 0.3,\ 0.0\} = 0.3.$)

On the gound of formula (14), based on the elements of these vectors the following matrix F can be built:

$$F = \begin{pmatrix} 0.9 & 1.0 & 0.8 \\ 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{pmatrix}.$$

(For example: element $F_{13} = Q_1^3 = \min\{1,\ 1 - p_2^{(1)} + q_2^{(3)}\} = \min\{1,\ 1 - 1 + 0.80\} = 0.8.$)

Passing to step 3° of the algorithm the matrices L'' and D'' should be created. Sorting out (in a descending order) the columns of the matrices RL' and RD', we obtain

$$L'' = \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.3 & 0.9 \\ 0.2 & 0.2 & 0.3 \end{pmatrix} \quad \text{and} \quad D'' = \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 \\ 0.8 & 1.0 & 0.8 \end{pmatrix}.$$

On the basis of the foregoing calculations it is possible to determine the matrix of assignments (truth losses) A. For example element A_{11} is calculated in the following way (see (20)):

$$A_{11} = 1 - (0.9 \cdot \tfrac{1}{2} \cdot \tfrac{1}{3}(1 + 1 + 1)) = 1 - 0.45 = 0.55.$$

Calculating the other elements in a similar way one obtains

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.55 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix} \end{array}.$$

Applying the VAM method, assuming in this situation the assignment according to the order of rows and columns, we locate facility 1 at place 2. In accordance with step 6° of the algorithm, deleting the columns and rows in the matrices RL' and RD', corresponding to the layout, we obtain

$$RL' = \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{pmatrix} 1.0 & 0.3 \\ 0.3 & 1.0 \end{pmatrix} \end{array}, \qquad RD' = \begin{array}{c} \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{pmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{pmatrix} \end{array}.$$

(Notice that in the formula (21) L' and D' denote the output, not reduced, matrices.)

Since 2 facilities are still left we pass to step 3° of the algorithm and create the matrices L'' and D'' through sorting out:

$$L'' = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{pmatrix} 1.0 & 1.0 \\ 0.3 & 0.3 \end{pmatrix} \end{array}, \qquad D'' = \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{pmatrix} 1.0 & 1.0 \\ 0.8 & 0.8 \end{pmatrix} \end{array}.$$

We determine a new matrix of the assignments A. For example,

$$A_{23} = 1 - 1 \cdot \tfrac{1}{2} \cdot 1 \cdot \{(1 - 0.2 + 1) \wedge 1\} + \tfrac{1}{2}(1 + 1) = 0.0.$$

Calculating all elements in this way we obtain

$$A = \begin{array}{cc} & \begin{array}{cc} 1 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{pmatrix} 0.0 & 0.0 \\ 0.05 & 0.0 \end{pmatrix} \end{array}.$$

Based on the VAM method we select the layout $2 \rightarrow 1$. Since only one element has been left out we locate it at the remaining free place i.e. $3 \rightarrow 3$.

We obtain the layout system presented in the Figure 1.

It is very easy to prove that this solution is optimum. Applying the formulae (16) and (17) to the system we obtain $O(p) = 1$ and $Q(p) = 1$ and hence $G(p) = 1$, $p = (2, 1, 3)$.
It is also very easily noticed that it is not the only solution than can be obtained by applying the proposed algorithm.

It is possible to use our 'maximal mean truth value' theorem to find the upper bound on the $O(p)$ values on the basis of L and D matrices. The matrix F allows one to easily find the upper bound on $Q(p)$. These two values can be used to test the optimality of a given solution in a general case.

Of course the test of the algorithm properties requires extended studies. From preliminary investigations by the author, performed by means of a 8-bit microcomputer, it resulted that (without a large scope of the problem) the proposed approach is very promising. (For 10 tested problems with casually

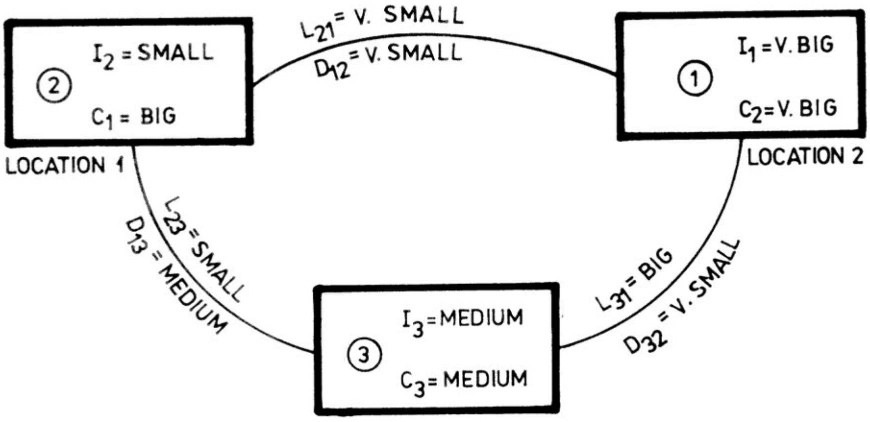

Fig. 1. The obtained layout with values of the linguistic variables.

defined data and $n = 3$ or 4 all solutions were optimal.) Systematic studies on the algorithm properties will be continued. The results will be presented in a separate paper.

6. Concluding remarks

The proposed approach widens the range of possibilities of applying the optimization rules used in the FLP in the case of non-precisely or 'fuzzy' formulated data.

The analysis of works dealing with the FLP in a 'classical' way implies that the situations which are not too precisely defined for various reasons) occur very often in practice (Gavett and Plyter [2], Bonney and Williams [1]). Certainly, it is necessary to investigate the properties of the proposed algorithm in detail. The investigation of the relations between traditional concepts and the fuzzy approach, as well as of the effects of the way of defining the fuzzy sets applied in the estimation of the task, individual parameters on the solution quality seem particularly interesting. The correlation of the proposed approach with the concepts resulting from investigations on expert systems – particularly in the range of the ways of formulation of unprecisely defianed data on the basis of expert group opinion – is thought to be profitable (Yager, [11], Freksa [4]).

Appendix 1

The course of calculations in the VAM method (Lis and Santarek [6]):

(a) The differences between the two smallest elements are calculated for each row and each column of the matrix:

$$R_i = A_{i,p_2} - A_{i,p_1}, \quad i = 1, 2, \ldots, n,$$

where $A_{i,p_2} \geqslant A_{i,p_1}$ and these are the two smallest elements in row i;

$$C_p = A_{i2,p} - A_{i1,p}, \quad p = 1, 2, \ldots, n,$$

where $A_{i2,p} \geqslant A_{i1,p}$ and these are the two smallest elements in column p.

(b) A row (or a column) is chosen to which the greatest of the calculated differences corresponds; it is the row i_k when

$$R_{i_k} = \max_{i,p=1,\ldots,n} \{R_i, C_p\},$$

or the column p_k when

$$C_{p_k} = \max_{i,p=1,\ldots,n} \{R_i, C_p\}.$$

(c) In the chosen row i_k (or column p_k) of the matrix A the minimum element is sought. This is an element from the column p_k when

$$A_{i_k,p_k} = \min_{p=1,\ldots,n} \{A_{i_k,p}\},$$

or an element of the row i_k when

$$A_{i_k,p_k} = \min_{i=1,\dots,n} \{A_{i,p_k}\}.$$

The facility corresponding to row i_k is located at place p_k. In the proposed method, at each step the matrix A lessens and consists of different elements – thus for a given matrix A the calculation cycle finishes after the point (c) has been performed.

Appendix 2

Theorem ('of the maximum mean truth'). *Let $p_1, p_2, \dots, p_n$ denote the set of 'truth values' of the statements $A_1, A_2, \dots, A_n$ and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$, $q_1, q_w, \dots, q_n$ denote the set of 'truth values' of the statements $B_1, B_2, \dots, B_n$ and $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n$ with*

$$p_i, q_j \in [0, 1].$$

Let $A_i \tilde{\Rightarrow} B_j$ denote a generalized implication and $t_{ij} = \min(1, 1 - p_i + q_j)$ the truth value of this implication (Tsukamoto [10]). Then the following is true:

$$\max_{\pi \in P} \left(\frac{1}{n} \sum_{i=1}^{n} t_{i,\pi_i} \right) = \frac{1}{n} \sum_{i=1}^{n} t_{ii},$$

where P is the set of all permutations of the natural numbers $1, 2, \dots, n$ and π_i the i-th element of permutation π.

That means that the greatest 'mean truth value' in the set of implications $\{A_i \tilde{\Rightarrow} B_{\pi_i}\}_{i=1}^{n}$ is obtained when the statements of A and B types are ordered respectively in a nonincreasing way according to the estimates of the truth of p_i and q_j, i.e. $A_1 \tilde{\Rightarrow} B_1$. $A_2 \tilde{\Rightarrow} B_2, \dots, A_n \tilde{\Rightarrow} B_n$.

Proof. Given the sequences of statements of A and B types ordered so that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ and, for a given pair j, k, $q_j \geqslant q_k$ if $j < k$, let us consider the sequence of statements B' defined as

$$B'_i = B_i \text{ for } i \neq j, k \quad \text{and} \quad B'_j = B_k, \; B'_k = B_j.$$

The difference between the 'mean truth' of both systems $\{A_i \tilde{\Rightarrow} B_{\pi_i}\}_{i=1}^{n}$ and $\{A_i \tilde{\Rightarrow} B_{\pi'_i}\}_{i=1}^{n}$ (π and π' denote permutations of B and B' sequences respectively) can be expressed as

$$\begin{aligned} T - T' &= \frac{1}{n} \sum_{i=1}^{n} t_{i\pi_i} - \frac{1}{n} \sum_{i=1}^{n} t_{i\pi'_i} \\ &= \frac{1}{n}(t_{jj} + t_{kk}) - \frac{1}{n}(t_{jk} + t_{kj}) \\ &= \frac{1}{n}[\min(1, 1 - p_j + q_j) + \min(1, 1 - p_k + q_k) \\ &\qquad - \min(1, 1 - p_j + q_k) - \min(1, 1 - p_k + q_j)]. \end{aligned}$$

One should prove that the expression in square brackets is $\geqslant 0$. In order to do this let us separately consider the sums $t_{jj} + t_{kk} = S_1$ and $t_{jk} + t_{kj} = S_2$.

Notice that

$$S_1^{(1)} = 2 \qquad \text{if } q_j \geqslant p_j \text{ and } q_k \geqslant p_k, \tag{A1}$$

$$S_1^{(2)} = 2 - p_k + q_k \qquad \text{if } q_j \geqslant p_j \text{ and } q_k \leqslant p_k, \tag{A2}$$

$$S_1^{(3)} = 2 - p_j + q_j \qquad \text{if } q_j \leqslant p_j \text{ and } q_k \geqslant p_k, \tag{A3}$$

$$S_1^{(4)} = 2 - (p_k + p_j) + (q_k + q_j) \quad \text{if } q_j \leqslant p_j \text{ and } q_k \leqslant p_k, \tag{A4}$$

whereas

$$S_2^{(1)} = 2 \qquad \text{if } q_k \geqslant p_j \text{ and } q_j \geqslant p_k, \tag{A1$'$}$$

$$S_2^{(2)} = 2 - p_k + q_j \qquad \text{if } q_j \leqslant p_k \text{ and } q_k \geqslant p_j, \tag{A2$'$}$$

$$S_2^{(3)} = 2 - p_j + q_k \qquad \text{if } q_j \geqslant p_k \text{ and } q_k \leqslant p_j, \tag{A3$'$}$$

$$S_2^{(4)} = 2 - p_k + p_j + q_k + q_j \quad \text{if } q_k \leqslant p_j \text{ and } q_j \leqslant p_k. \tag{A4$'$}$$

So the difference $S_1 - S_2$ can be presented in 16 different ways according to the conditions (1)–(4$'$). Let us assume that $j < k$ and hence

$$p_j \geqslant p_k \tag{A5}$$

and

$$q_j \geqslant q_k; \tag{A6}$$

then

$S_1 - S_2 =$ 1° $S_1^{(1)} - S_2^{(1)} = 2 - 2 = 0,$

OR 2° $S_1^{(1)} - S_2^{(2)} = p_k - q_j = 0,\ p_k \geqslant q_j$ and $p_k \leqslant q_j$ (from A1, A2$'$, A5, A6)

OR 3° $S_1^{(1)} - S_2^{(3)} = p_j - p_k \geqslant 0,\ p_j \geqslant p_k$ (from A1, A3$'$, A5, A6)

OR 4° $S_1^{(1)} - S_2^{(4)} = p_k + p_j - (q_k + q_j) = 0,$ (from A1, A4$'$, A5, A6)
$p_k = q_k = q_j = p_j$

OR 5° $S_1^{(2)} - S_2^{(1)} = q_k - p_k = 0,\ p_k \geqslant q_k$ and $p_k \leqslant q_k$ (from A2, A1$'$, A5, A6)

OR 6° $S_1^{(2)} - S_2^{(2)} = q_k - q_j = 0,\ q_k \leqslant q_j$ and $q_k \geqslant q_j$ (from A2, A2$'$, A5, A6)

OR 7° $S_1^{(2)} - S_2^{(3)} = p_j - p_k \geqslant 0,\ p_j \geqslant p_k$ (from A2, A3$'$, A5, A6)

OR 8° $S_1^{(2)} - S_2^{(4)} = p_j - q_j = 0,\ p_j \geqslant q_j$ and $p_j \leqslant q_j$ (from A2, A4$'$, A5, A6)

OR 9° $S_1^{(3)} - S_2^{(1)} = q_j - p_j = 0,\ q_j \geqslant p_j$ and $q_j \leqslant p_j$ (from A3, A1$'$, A5, A6)

OR 10° $S_1^{(3)} - S_2^{(2)} = p_k - p_j = 0,\ p_k \geqslant p_j$ and $p_j \geqslant p_k$ (from A3, A2$'$, A5, A6)

OR 11° $S_1^{(3)} - S_2^{(3)} = q_j - q_k \geqslant 0,\ q_j \geqslant q_k$ (from A3, A3$'$, A5, A6)

OR 12° $S_1^{(3)} - S_2^{(4)} = p_k - q_k = 0,\ p_k \geqslant q_k$ and $p_k \leqslant q_k$ (from A3, A4$'$, A5, A6)

OR 13° $S_1^{(4)} - S_2^{(1)} = q_k + q_j - (p_k + p_j) = 0,$ (from A4, A1$'$, A5, A6)
$p_k = q_k$ and $p_j = q_j$

OR 14° $S_1^{(4)} - S_2^{(2)} = q_k - p_j = 0,\ q_k \geqslant p_j$ and $q_k \leqslant p_j$ (from A4, A2$'$, A5, A6)

OR 15° $S_1^{(4)} - S_2^{(3)} = q_j - p_k \geqslant 0,\ q_j \geqslant p_k$ (from A4, A3′, A5, A6)

OR 16° $S_1^{(4)} - S_1^{(4)} = 2 - 2 = 0.$

From the above list it can be noticed that for all possible situations $S_1 - S_2 \geqslant 0$, and hence the 'mean truth' in the set $\{A_i \tilde{\Rightarrow} B_{\pi_i}\}_{i=1}^n$ is greater than that in the set $\{A_i \tilde{\Rightarrow} B_{\pi'_i}\}_{i=1}^n$. In the same way inequality $S_1 - S_2 \geqslant 0$ can be proved for $j > k$.

Since an optional permutation of the set B can be represented as a series of permutations of B' type, where only one pair of elements has been changed and each of these pairs preserves the inequality $S_1 - S_2 \geqslant 0$ (Gavett and Plyter [2]), the theorem of 'the maximum mean truth' is true.

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